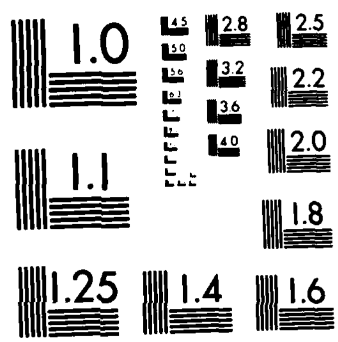
AD-A147 126 ON THE POWER FUNCTIONS OF TEST STATISTICS IN ORDER RESTRICTED INFERENCE(U) IOWA UNIV IOWA CITY DEPT OF STATISTICS AND ACTURIAL SCIENCE.. H MUKERJEE ET AL. OCT 84 TR-107 N00014-80-C-0321 1/1

UNCLASSIFIED F/G 12/1 NL

END




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A147 126

ON THE POWER FUNCTIONS OF TEST STATISTICS

IN ORDER RESTRICTED INFERENCE(1)

Hari Mukerjee
Division of Statistics
University of California-Davis
Davis, California 95616

Tim Robertson(2)
Department of Statistics and Actuarial Science
The University of Iowa
Iowa City, Iowa 52242

F. T. Wright
Department of Mathematics and Statistics
University of Missouri-Rolla
Rolla, Missouri 65401

Technical Report No. 107

Department of Statistics and Actuarial Science
The University of Iowa
Iowa City, Iowa 52242

October 1984



(1) This research was sponsored by the Office of Naval Research under ONR contracts N00014-80-C0321 and N00014-80-C0322.

(2) Part of this author's research was done while he was a visiting professor at the University of California-Davis.

# ON THE POWER FUNCTIONS OF TEST STATISTICS IN ORDER RESTRICTED INFERENCE

Hari Mukerjee, Tim Robertson, and F. T. Wright

A-1

SUMMARY

We study the power functions of both the likelihood ratio and contrast statistics for detecting a totally ordered trend in a collection of means of normal populations. Monotonicity properties are found and both radial limits and limits along lines parallel to the cone of points satisfying the trend are examined. An optimal contrast test for testing a trend as a null hypothesis is derived.

AMS 1980 subject classifications: Primary 62F03; Secondary 62F04.

Key words and phrases: Order restricted tests, isotonic inference, power, likelihood ratio tests, contrast tests.

— A —

INTRODUCTION. We consider the powers of statistical tests for detecting a trend in a collection of population parameters. The statistical literature contains a number of such tests and a detailed summary of this research up to about 1971 is given in Barlow et al. (1972). More recent summaries are given in Bartholomew (1983), Lee (1983) and Robertson (1984). We restrict our attention to the case in which the parameters, $\mu_1, \mu_2, \ldots, \mu_k$, are the means of normal populations and the trend restriction requires them to be totally ordered. To be specific we consider the trend $H_1: \mu_1 \le \mu_2 \le \cdots \le \mu_k$. One approach to detecting such a trend is to test homogeneity, $H_0: \mu_1 = \mu_2 = \cdots = \mu_k$, versus $H_1 - H_0$, i.e., $H_1$ holds with $\mu_1 < \mu_k$ (cf. Bartholomew (1959 a,b; 1961)). Even in the case of normal means the results concerning the powers of these restricted tests are very exiguous and consist primarily of comparisons with the powers of other tests, such as the unrestricted tests of $H_0$ versus $H_1': \mu_i \ne \mu_j$ for some $i \ne j$. In fact, as far as we can determine, the first mention of the fact that Bartholomew's tests are unbiased occurs in Robertson and Wright (1982). The biases of other restricted tests are examined in Dykstra and Robertson (1983).

Assuming independent random samples from normal populations, Bartholomew (1959 a,b; 1961) studied the likelihood ratio tests (LRTs) for $H_0$ versus $H_1 - H_0$ assuming in one case that the population variances are known and in the other that they are unknown but equal (partially ordered trends were also considered). We focus attention on the case of known variances. Results concerning the unknown variances case follow by conditioning arguments in the last section. Implementation of Bartholomew's test procedures can be difficult for $k > 5$ if the so-called weights are not equal. (For the known variances case, the weights are the precisions, $n_i/\sigma_i^2$, of the sample means as estimates of the population means.) This difficulty is mainly due to the fact that the level probabilities involved in the null hypothesis distribution of the test statistic are extremely difficult to compute in such cases. This theory is discussed at length in Chapter 3 of Barlow et al. (1972). Robertson and Wright (1983) proposed an approximation for the level probabilities for the case of total order and Pillers et al. (1984) gives a computer routine for implementing this approximation.

Partly because of the difficulty involved in applying Bartholomew's procedures, several researchers, including Abelson and Tukey (1963) and Schaafsma and Smid (1966), considered tests based upon contrasts (cf. Section 4.2 of Barlow et al. (1972)). Denoting the sample means by $\overline{X}_i$, $1 \le i \le k$, these contrast tests are based upon statistics of the form $T_c = \sum_{i=1}^{k} c_i\overline{X}_i$ where $c = (c_1, c_2, \cdots, c_k)$ is a vector of predetermined constants $(\sum_{i=1}^{k} c_i = 0)$. One attraction of these constrasts is the fact that their distribution at any point $\mu = (\mu_1, \mu_2, \cdots, \mu_k) \in R^k$ is normal. With $\mu_0, \nu \in R^k$ fixed $(\sum_{i=1}^{k} \nu_i = 0)$, the uniformly most powerful (UMP) test of $H_0'\colon \mu = \mu_0$ versus $H_1'\colon \mu \in \{\mu_0 + b\nu;\ b > 0\}$ rejects $H_0'$ for large values of $T_\nu$ (use the Neyman-Pearson Theorem and note that this test is UMP for fixed $b > 0$). Since this test does not depend on $\mu_0 \in H_0$ it is not surprising that contrast tests are very powerful in some subregion of the alternative. However, even for moderate $k$, there are other subregions of the alternative where the power of the contrast test does not compare favorably with the power of Bartholomew's LRT. While the LRT is not most powerful at any particular point, it does maintain a more uniformly reasonable power over all of $H_1$. One explanation of this fact is given in Theorem 4.3 of Barlow et al. (1972), which can be interpreted to say that the LRT is based on an "adaptive" contrast statistic. In other words, the parameters are estimated from the data and then the contrast coefficients are chosen so that the test has a relatively high power at the estimated point.

In the references cited earlier, Abelson and Tukey (1963) and Schaafsma and Smid (1966) derived optimal contrast tests. In the former they obtained the contrast coefficients which maximize the minimum power over all points equidistant from $H_0$ and in the latter those that minimize the maximum loss of power as compared to the most powerful test in a restricted class of procedures. The powers of these optimal contrast tests are compared with that of the LRT in Section 4.2 of Barlow et al. (1972). Their conclusion is that, while it is difficult to improve on the LRT, for a total order and small $k$ these contrast tests provide viable alternatives to the LRT.

In the above approach to detecting a trend the hypothesis of homogeneity is a "dummy" hypothesis. Control of the $\alpha$ level provides protection against confirming the trend when it is not present. Robertson and Wegman (1978) and Robertson (1978) studied LR procedures for testing a trend as a null hypothesis. The $\alpha$-level of these tests controls the probability of denying the trend when it is present. The null hypothesis distributions of the LRT statistics involve the same level probabilities that complicate the use of Bartholomew's test. In Section 4 we derive an optimal contrast test for $H_1$ versus $H_2$: $\mu_i > \mu_{i+1}$ for some $i$.

In Section 3, we study the power functions of the LRTs for testing $H_0$ versus $H_1 - H_0$ and for $H_1$ versus $H_2$. The power functions for the latter tests are more complicated and, in a sense, more interesting. Some preliminary results for studying the powers of the LRTs are developed in Section 2. The competing contrast tests are discussed in Section 4.

Throughout this paper $\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_k$ denote the sample means of independent random samples with $\bar{X}_i \sim n(\mu_i, \sigma_i^2/n_i)$, $n_i$ the size of the $i^{\underline{th}}$ sample, and $\sigma_i^2$ the variance of the $i^{\underline{th}}$ population. Assume that the variances are known and set $w_i = n_i/\sigma_i^2$ for $i = 1,2,\cdots,k$. Let $H_0$ and $H_1$ denote the hypotheses specified earlier as well as the corresponding subsets in $R^k$. The set, $H_0$, is a subspace and $H_1$ is a closed, convex cone. Let $(\cdot,\cdot)_W$ denote the inner product on $R^k$ defined by $(x,y)_W = \sum_{i=1}^k w_i x_i y_i$ and let $\|\cdot\|_W$ denote the corresponding norm. Bartholomew's test of $H_0$ versus $H_1 - H_0$ rejects $H_0$ for large values of

$$(1.1) \qquad T_{01} = -2 \ln \Lambda = \sum_{i=1}^k w_i(\bar{\mu}_i - \hat{\mu})^2 = \|\bar{\mu} - \hat{\mu} e_k\|_W^2$$

where $\Lambda$ denotes the likelihood ratio, $\hat{\mu} = \sum_{i=1}^k w_i \bar{X}_i / \sum_{i=1}^k w_i$, $e_k$ is a k-dimensional vector of ones and $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \cdots, \bar{\mu}_k)$ minimizes $\sum_{i=1}^k w_i (g_i - \bar{X}_i)^2$ subject to $g \in H_1$. In other words, $\bar{\mu}$ is the projection of $\bar{X} = (\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_k)$ onto $H_1$ with respect to the distance $d(x,y) = \|x-y\|_W$. We will also denote this projection by $E_W(\bar{X} \mid H_1)$. With the closed, convex cone $H_1 \cap \{x : \sum_{i=1}^k x_i w_i = 0\}$ denoted by $C_{01}$, Theorem 1.5 of Barlow et al. (1972) can be used to show that $E_W(\bar{X} \mid C_{01}) = E_W(\bar{X} \mid H_1) - \hat{\mu}$ and so $T_{01} = \|E_W(\bar{X} \mid C_{01})\|_W^2$. Therefore, an acceptance region for $T_{01}$ can be written as $\{x \in R^k : \|E_W(x \mid C_{01})\|_W^2 \le t\}$ for some $t > 0$.

The LRT of $H_1$ versus $H_2$ rejects for large values of

(1.2) $\quad T_{12} = -2 \ln \Lambda = \sum_{i=1}^{k} w_i(\bar{X}_i - \bar{\mu}_i)^2 = \|\bar{X} - E_w(\bar{X} \mid H_1)\|_w^2.$

If $C_{12}$ denotes the dual of $H_1$ (also called the polar or conjugate of $H_1$), which is a closed convex cone whose definition is given in the next section, then Theorem 1.5 of Barlow et al. (1972) shows that $E_w(x \mid C_{12}) = x - E_w(x \mid H_1)$, and so $T_{12} = \|E_w(\bar{X} \mid C_{12})\|_w^2$. The acceptance regions for $T_{12}$ are of the form $\{x \in R^k : \|E_w(x \mid C_{12})\|_w^2 \le t\}$ with $t > 0$.

In Section 3 we consider the question of unbiasedness for $T_{01}$ and $T_{12}$ as well as the radial behavior of their power functions, that is, their behavior on the sets $\{\delta\mu;\ \delta \ge 0\}$ for various $\mu$. The behavior of these power functions in other directions is also discussed. Robertson and Wright (1982) considered the relation on $R^k$, $x \lesssim y$ provided $y-x \in H_1$. They showed that $T_{01}$ and its power function are isotone, and $T_{12}$ and its power function are antitone with respect to $\lesssim$. This implies that if $\mu \in H_1$, then the power of $T_{01}$ $(T_{12})$ is nondecreasing (nonincreasing) in $\delta$ on $\{\delta\mu : -\infty < \delta < \infty\}$. Their results concerning the stronger relation $\ll$ show that the power of $T_{01}$ is monotone in a larger set of directions, but these techniques do not apply to $T_{12}$ because it is not antitone with respect to $\ll$. However, because of the strong similarities in the acceptance regions for the tests $T_{01}$ and $T_{12}$, one might conjecture that $T_{12}$ is also monotone in this larger set of directions. Using the geometric arguments of Section 2, this is shown to be the case.

2. SOME PROBABILITY INEQUALITIES. The probability inequalities derived in this section will be used in the discussion of the montonicity of the power functions of $T_{01}$ and $T_{12}$. Using the techniques in Bartholomew (1961), one can, at least in principle, obtain analytic expressions for these power functions in terms of several multiple integrals, but, as functions of the distance of a mean vector from $H_0$ and its direction, they are extremely intractable even in the case of equal weights and $k = 3$. (See Section 3 for further discussion.) We have resorted to geometric arguments involving integrals of symmetric unimodal densities over convex sets that have certain symmetry properties. In a sense, the results obtained are generalizations of Anderson's (1955) work on similar integrals over symmetric (about the origin) convex sets, but in the present work, the statistics are projections on closed, convex cones which are not subspaces, and thus the sets involved have only partial symmetry.

The basic idea is the following. Let $P_\mu(\cdot)$ denote the $n(\mu,I)$ probability distribution on $R^k$ for each $\mu \in R^k$, and note that $P_\mu(A) = P_0(A-\mu)$ for all measurable $A \subset R^k$. If $A$ is the acceptance region for one of the tests considered here, then $A$ is closed and convex. If $S_\mu$ is the subspace generated by a mean vector $\mu$ and $S_\mu^\perp$ is the orthogonal complement of $S_\mu$ in $R^k$, then, for some directions of $\mu$, it is possible to decompose $A$ into disjoint subsets $A'$ and $A''$, with $A'$ a closed, convex set symmetric about $S_\mu^\perp$ and $A''$ on one side of $S_\mu^\perp$. This will enable us to prove the monotonicity of $P_{\delta\mu}(A) = P_0(A-\delta\mu)$ in $\delta \geq 0$ for such directions.

Because we anticipate the application of the results of this section to more general types of cones than those considered in this paper, and because we believe the results concerning projections on closed, convex cones in a real Hilbert space are of interest in themselves, we consider a more general framework than is needed for this paper.

Let $H$ denote a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. If $C$ is a closed, convex cone in $H$ and $x \in H$, then $E(x|C)$ will denote the unique projection of $x$ onto $C$, i.e., $E(x|C)$ is the unique element of $C$ which minimizes $\|x-y\|$ as $y$ ranges over $C$. Theorem 7.8 of Barlow et al. (1972) characterizes $E(x|C)$ as follows:

(2.1) $E(x|C) \in C$, $(x-E(x|C), E(x|C)) = 0$, and $(x-E(x|C),y) \le 0$ for all $y \in C$.

It follows from (2.1) that $E(ax|C) = aE(x|C)$ for $a \ge 0$ and $x \in H$, and that

(2.2) $(x,E(x|C)) = (E(x|C),E(x|C)) = \|E(x|C)\|^2$ for $x \in H$.

The dual of $C$, which is denoted by $C^*$, is defined by $C^* = \{x \in H: (x,y) \le 0 \text{ for all } y \in C\}$. Clearly, $C^*$ is a closed, convex cone and using the definition of $C^*$, (2.1), and (2.2), we see that

(2.3) $E(x|C^*) = x - E(x|C)$ and $\|E(x|C^*)\|^2 = \|x\|^2 - \|E(x|C)\|^2$.

In the Appendix it is shown that

(2.4) $(C^*)^* = C$.

Throughout this section $C$ will denote a closed, convex cone in $H$, $C^*$ its dual, and $A$ the closed set $\{x \in H: \|E(x|C)\| \le t\}$ for some $t > 0$. For $\mu \in H$ let $C_\mu = \{b\mu: b \ge 0\}$, $S_\mu = \{b\mu: -\infty < b < \infty\}$, and let $S_\mu^\perp$ be the orthogonal complement of $S_\mu$.

We repeat for the reader's convenience a result from Robertson and Wegman (1978) which will be used several times in the sequel.

Lemma 2.1. If $x \in C$, then for any $y \in H$, $\|x+y - E(x+y \mid C)\| \le \|y - E(y \mid C)\|$ or, equivalently, $\|E(x+y \mid C^*)\| \le \|E(y \mid C^*)\|$.

Next, we state several lemmas which are proved in the Appendix. Let $-D = \{-x : x \in D\}$.

Lemma 2.2. Suppose $\mu \in C \cup (-C^*)$. For $x \in H$ and $0 \le b \le 2$,

$$\|E(x - bE(x-\mu_0 \mid C_\mu) \mid C)\| \le \|E(x \mid C)\| \quad \text{for all} \quad \mu_0 \in S_\mu^\perp. \tag{2.5}$$

So, if $(\mu,\mu_0) = 0$ and $y \in A-\mu_0$, then $y-bE(y \mid C_\mu) \in A-\mu_0$ for $0 \le b \le 2$.

Lemma 2.3. For $x,y \in H$,

$$\|E(x+y \mid C)\| \le \|E(x \mid C)+E(y \mid C)\| \le \|E(x \mid C)\| + \|E(y \mid C)\|.$$

Using Lemma 2.3, we see that $A$ is convex.

Lemma 2.4. Let $S$ be a closed subspace of $H$.

(a) If $S \subset C$, then $(x-E(x \mid C),v) = 0$ and $E(x-v \mid C) = E(x \mid C) - v$ for all $x \in H$ and $v \in S$.

(b) If $S \subset C$, then $E(E(x \mid C) \mid S) = E(x \mid S)$ for all $x \in H$.

(c) If $S \subset C$, then $E(x \mid C) - E(x \mid S) = E(x \mid C \cap S^\perp)$ for all $x \in H$.

(d) If $C \subset S$, then $E(E(x \mid S) \mid C) = E(x \mid C)$ for all $x \in H$.

The next result identifies the portion of $A$ that is symmetric about $S_\mu^\perp$. Define $A^+$ to be $\{x \in A : E(x \mid S_\mu) = b\mu$ for some $b \ge 0\}$ and $B$ to be $\{x - bE(x \mid S_\mu) : x \in A^+, 0 \le b \le 2\}$.

Theorem 2.5. If $\mu \in C \cup (-C^*)$ and $\mu_0$ is any vector in $H$ with $(\mu,\mu_0) = 0$, then $B-\mu_0 \subset A-\mu_0$, $B-\mu_0$ is symmetric about $S_\mu^\perp$, i.e., $x \in B-\mu_0$ implies $x - 2E(x \mid S_\mu) \in B-\mu_0$, and $B-\mu_0$ is closed and convex.

Proof. First assume the theorem is true with $\mu_0 = 0$. Then the first and the last conclusions follow immediately for arbitrary $\mu_0 \in H$. The second conclusion follows when $(\mu,\mu_0) = 0$ by noting that, when $B$ is symmetric about $S_\mu^\perp$, $x \in B-\mu_0$ implies $x+\mu_0 - 2E(x+\mu_0 \mid S_\mu)$ $= x+\mu_0 - 2E(x \mid S_\mu) \in B$, so that $x - 2E(x \mid S_\mu) \in B-\mu_0$. Thus it is suffic t to prove the theorem for $\mu_0 = 0$.

For $x \in A^+$, $E(x \mid C_\mu) = E(x \mid S_\mu)$ and so $B \subset A$ by Lemma 2.2. If $x \in B$, then there exists $y \in A^+$ and $b \in [0,2]$ with $x = y - bE(y \mid S_\mu)$. Thus by the linearity of $E(\cdot \mid S_\mu)$,

$$x - 2E(x|S_\mu) = y - bE(y|S_\mu) - 2(1-b)E(y|S_\mu) = y - (2-b)E(y|S_\mu) \in B.$$

Next, we show that $B$ is closed. Since $S_\mu$ is 1-dimensional it is closed, and so is $S_\mu^\perp$. Because $A^+$ is the intersection of the closed, convex sets $A$ and $\{x \in H: E(x \mid S_\mu) = b\mu, b \ge 0\}$, it is closed and convex also. Now $A^+ = \{x - bE(x \mid S_\mu): x \in A^+, 0 \le b \le 1\}$, and if we define $A^- = \{x - bE(x \mid S_\mu): x \in A^+, 1 \le b \le 2\}$, then $A^-$ is the reflection of $A^+$ across $S_\mu^\perp$. It is then easily verified that $A^-$ is closed and convex. Hence, $B = A^+ \cup A^-$ is closed.

Since $A^+$ and $A^-$ are both convex, to show that $B$ is convex, we need only show that $ax + (1-a)y \in B$ for $0 < a < 1$, $x \in A^+ \cap (A^-)^c$ and $y \in A^- \cap (A^+)^c$. Now $y - 2E(y \mid S_\mu) \in B$ and $E(y - 2E(y|S_\mu)|S_\mu) = -E(y|S_\mu)$

implies that $y - 2E(y \mid S_\mu) \in A^+$. Thus, $\nu = ax + (1-a)(y - 2E(y \mid S_\mu)) \in A^+$ and $\nu - bE(\nu \mid S_\mu) \in B$ for $0 \le b \le 2$. Now, $E(x \mid S_\mu) = c_x\mu$ and $E(y \mid S_\mu) = c_y\mu$ with $c_x > 0$ and $c_y < 0$. Then $E(\nu \mid S_\mu) = (ac_x - (1-a)c_y)\mu$ and $\nu - b(ac_x - (1-a)c_y)\mu \in B$ for all $0 \le b \le 2$. But $0 < -2(1-a)c_y < 2(ac_x - (1-a)c_y)$. Thus

$$ax + (1-a)y = \nu + 2(1-a)c_y\mu = \nu - [(-2(1-a)c_y)/(ac_x - (1-a)c_y)]E(\nu \mid S_\mu) \in B,$$

and the proof is completed.

For the remainder of this section, we take $H$ to be $R^k$ and at first consider the usual inner product. For $\mu \in R^k$ we let $P_\mu$ denote the $n(\mu, I)$ probability distribution on $R^k$.

<u>Theorem 2.6</u>. If $\mu \in R^k$ and $E$ is a closed, convex set in $R^k$ which is symmetric about $S_\mu^\perp$, then $P_{\delta\mu}(E)$ is nonincreasing in $\delta$ for $\delta \ge 0$.

<u>Proof</u>. We may assume that $\mu \ne 0$ and $\|\mu\| = 1$. Introduce in $R^k$ an orthonormal coordinate system, so that if $x \in R^k$ has coordinates $(x_1, x_2, \cdots, x_k)$, then $E(x \mid S_\mu) = (x_1, 0, \cdots, 0)$. Hence, for any measurable $D \subset R^k$,

$$P_0(D) = \int \Big( \int_{D(x_1)} \cdots \int \Pi_{i=2}^k \varphi(x_i)dx_i \Big) \varphi(x_1)dx_1 = \int g_D(x_1)\varphi(x_1)dx_1,$$

where $D(x_1) = \{(x_2, \cdots, x_k) : (x_1, x_2, \cdots, x_k) \in D\}$ is the $x_1$-section of $D$ and $\varphi$ is the standard normal density. From the symmetry assumption on $E$, $E(-x_1) = E(x_1)$, and because $E$ is convex and symmetric, $E(ax_1) \subset E(a'x_1)$ for $0 \le a' \le a$. So, $g_E$ is symmetric, nonnegative, and nondecreasing on $(-\infty, 0]$, and

$$P_{\delta\mu}(E) = P_0(E-\delta\mu) = \int g_E(x_1)\varphi(x_1-\delta)dx_1 = \int g_E(x_1+\delta)\varphi(x_1)dx_1.$$

Suppose $0 \le \delta \le \delta'$. Let

$$I = P_0(E-\delta\mu) - P_0(E-\delta'\mu) = \int[g_E(x_1+\delta) - g_E(x_1+\delta')]\varphi(x_1)dx_1.$$

Let $c = (\delta+\delta')/2$ and $b = (\delta'-\delta)/2$ and note that $c \ge b \ge 0$. Hence,

$$I = \int[g_E(x_1+c-b) - g_E(x_1+c+b)]\varphi(x_1)dx_1$$

$$= \int[g_E(y-b) - g_E(y+b)]\varphi(y-c)dy = \int h(y)\varphi(y-c)dy,$$

where $h(y) = g_E(y-b) - g_E(y+b)$ is antisymmetric and $h(y) \ge 0$ for $y \ge 0$. Thus,

$$I = \int_0^\infty h(y)[\varphi(y-c) - \varphi(y+c)]dy.$$

For $y \ge 0$ and $c \ge 0$, $(y+c)^2 \ge (y-c)^2$ and so $I \ge 0$. The proof is completed.

Remark. Replacing $\mu$ by $-\mu$ in Theorem 2.6, we see that $P_{-\delta\mu}(E)$ is nonincreasing in $\delta \ge 0$ also.

Theorem 2.7. If $\mu \in R^k$ and $D$ is a measurable subset of $R^k$ with $E(x \mid C_\mu) = 0$ for each $x \in D$, then $P_{\delta\mu}(D)$ is nonincreasing in $\delta \ge 0$.

Proof. We assume $\mu \ne 0$ and $\|\mu\| = 1$. We use the coordinate system and notation used in the proof of Theorem 2.6.

Because of the hypothesis on $D$, $D(x_1) = \emptyset$ and $g_D(x_1) = 0$ for all $x_1 > 0$. If $0 \le \delta \le \delta'$, then $P_0(D-\delta\mu) - P_0(D-\delta'\mu)$ $= \int_{(-\infty,0)} g_D(x_1)[\varphi(x_1-\delta) - \varphi(x_1-\delta')]dx_1$. Since $g_D(x_1) \ge 0$ and

$\varphi(x_1-\delta) \geq \varphi(x_1-\delta')$ for $x_1 \leq 0$ and $0 \leq \delta \leq \delta'$, the last integral is nonnegative. The proof is completed.

If $D$ and $D'$ are subsets of a linear space, we write $D \oplus D'$ to denote the direct sum, i.e., $D \oplus D' = \{x+y : x \in D, y \in D'\}$.

Theorem 2.8. Let $\mu \in R^k$, $\mu_0 \in S_\mu^\perp$, and let $S$ be a subspace in $R^k$ containing $C$. If $\mu \in (C \oplus S^\perp) \cup (-C^*)$, then $P_{\delta\mu}(A-\mu_0)$ is nonincreasing in $\delta \geq 0$, and, if $\mu \in C^*$, then $P_{\delta\mu}(A-\mu_0)$ is nondecreasing in $\delta \geq 0$.

Proof. First, consider the case $\mu \in C \cup (-C^*)$. Combining Theorems 2.5 and 2.6, we see that $P_{\delta\mu}(B-\mu_0)$ is nonincreasing in $\delta \geq 0$. If $x \in A \cap B^c-\mu_0$ then $E(x \mid C_\mu) = 0$, and, applying Theorem 2.7, $P_{\delta\mu}(A \cap B^c-\mu_0)$ is nonincreasing in $\delta \geq 0$. With $\mu \in C \cup (-C^*)$ and $\nu \in S^\perp$, $P_{\delta(\mu+\nu)}(A-\mu_0) = P_{\delta\mu}(A-\mu_0-\delta\nu)$. The first conclusion will be established by showing $A-\delta\nu = A$. Now, $A-\delta\nu = \{x-\delta\nu : x \in A\}$ $= \{x-\delta\nu : \|E(x \mid C)\| \leq t\} = \{y : \|E(y+\delta\nu \mid C)\| \leq t\}$. Applying part (d) of Lemma 2.4, the linearity of $E(\cdot \mid S)$, and the fact that $E(\delta\nu \mid S) = 0$, we see that

$$E(y+\delta\nu \mid C) = E(E(y+\delta\nu \mid S) \mid C) = E(E(y \mid S) \mid C) = E(y \mid C).$$

So $A-\delta\nu = \{y : \|E(y \mid C)\| \leq t\} = A$.

For the second conclusion, we assume that $\mu \in C^*$ and $0 \leq \delta \leq \delta'$. Since $(\delta'-\delta)\mu \in C^*$, by Lemma 2.1, $\|E(x+\delta'\mu \mid C)\| \leq \|E(x+\delta\mu \mid C)\|$. Thus $x+\delta\mu \in A$ implies that $x+\delta'\mu \in A$ or $A+\delta'\mu \subset A-\delta\mu$, so that $A-\mu_0-\delta'\mu \subset A-\mu_0-\delta\mu$. Hence, $P_{\delta'\mu}(A-\mu_0) = P_0(A-\mu_0-\delta'\mu) \geq P_0(A-\mu_0-\delta\mu)$ $= P_{\delta\mu}(A-\mu_0)$. The proof is completed.

Theorem 2.8 will be used to study the monotonicity of the power functions of $T_{01}$ and $T_{12}$ in the case of equal weights, i.e., $w_1 = w_2 = \cdots = w_k$. The analogous results for unequal weights will be established next. Let $w_i > 0$ for $i = 1,2,\cdots,k$ and let $W$ be the $k\mathrm{x}k$ diagonal matrix with $W_{ii} = w_i$, $i = 1,\cdots,k$. Consider the inner product, $(\cdot,\cdot)_W$, and norm, $\|\cdot\|_W$, on $R^k$ which were defined in the Introduction. For $C$ a closed, convex cone in $R^k$ (closed in the topology induced by $\|\cdot\|_W$), let $C^{*W}$ denote the dual of $C$ and $E_W(\cdot \mid C)$ denote the projection onto $C$ with respect to $(\cdot,\cdot)_W$; for $S$ a subspace in $R^k$, let $S^{\perp W}$ denote its orthogonal complement with respect to $(\cdot,\cdot)_W$; for fixed $t > 0$, set $A_W = \{x \in R^k : \|E_W(x \mid C)\|_W \leq t\}$; and for $\mu \in R^k$, let $P_{\mu,W}$ denote the normal probability distribution on $R^k$ with mean $\mu$ and covariance matrix $W^{-1}$. If $W = I$, then we will omit the subscript or superscript, except for emphasis.

We now establish two identities involving dual cones and projections with respect to $(\cdot,\cdot)_I$ and $(\cdot,\cdot)_W$. They can easily be generalized to the case in which $W$ is a positive definite matrix or an invertible positive operator on a real Hilbert space. In either case, the inner product $(x,y)_W = (x,Wy)_I$.

Let $W^{1/2}$ denote the unique positive square root of $W$ and let $O$ be any $k\mathrm{x}k$ orthogonal matrix. The matrix $O$ plays no essential role, however, a judicious choice, such as a generalized Helmert transformation in the case of a totally ordered trend, may help identify the transformed parameters and visualize the transformed cone. Let $F = OW^{1/2}$ and note that $F$ is invertible. For $x,y \in R^k$, it is easily verified that $(x,y)_W = (Fx,Fy)$, $\|x\|_W = \|Fx\|$, and $(x,y)_W/(\|x\|_W\|y\|_W) = (Fx,Fy)/(\|Fx\|\|Fy\|)$, with $Fx \neq 0$ and $Fy \neq 0$ in the latter case. *Since* $F$ *is linear and* invertible $FC$ is also a closed, convex cone and $C = F^{-1}(FC)$. The following lemma is proved in the Appendix.

<u>Lemma 2.9</u>. $FC^{*W} = (FC)^{*I}$ and $E(Fx \mid FC) = FE_W(x \mid C)$ for all $x \in R^k$.

We now prove the following generalization of Theorem 2.8:

<u>Theorem 2.10</u>. Let $\mu \in R^k$, $\mu_0 \in S_\mu^\perp$, and let $S$ be a subspace in $R^k$ containing $C$. If $\mu \in (C \oplus S^{\perp W}) \cup (-C^{*W})$, then $P_{\delta\mu,W}(A_W - \mu_0)$ is nonincreasing in $\delta \geq 0$, and, if $\mu \in C^{*W}$, then $P_{\delta\mu,W}(A_W - \mu_0)$ is nondecreasing in $\delta \geq 0$.

<u>Proof</u>. If $X$ is distributed as $P_{\mu,W}$ then $Y = FX$ has a $\eta(F\mu, I)$ distribution. So $P_{\delta\mu,W}(A_W - \mu_0) = P_{\delta F\mu,I}(F(A_W - \mu_0)) = P_{\delta F\mu,I}(FA_W - F\mu_0)$, and we will apply Theorem 2.8 to the latter term with $\mu$, $\mu_0$, $S$, $C$, and $A$ replaced by $F\mu$, $F\mu_0$, $FS$, $FC$, and $FA_W$, respectively. Note that $(F\mu, F\mu_0) = (\mu, \mu_0)_W = 0$, $FC \subset FS$, $(FC \oplus (FS)^{\perp I}) \cup (-(FC)^{*I})$ $= F[(C \oplus S^{\perp W}) \cup (-C^{*W})]$ and so $\mu \in (C \oplus S^{\perp W}) \cup (-C^{*W})$ implies $F\mu \in (FC \oplus (FS)^{\perp I}) \cup (-(FC)^{*I})$. Now, using Lemma 2.9,

$$FA_W = \{Fx : \|E_W(x \mid C)\|_W \leq t\} = \{Fx : \|FE_W(x \mid C)\|_I \leq t\}$$

$$= \{Fx : \|E(Fx \mid FC)\| \leq t\}$$

Thus, by Theorem 2.8, $P_{\delta\mu,W}(A_W - \mu_0) = P_{\delta F\mu,I}(FA_W - F\mu_0)$ is nonincreasing in $\delta \geq 0$.

The proof of the generalization of the second conclusion of Theorem 2.8, which is similar, is omitted.

3. LIKELIHOOD RATIO TEST. We now apply the results in Section 2 to study the power functions of the LRTs of $H_0$ versus $H_1 - H_0$ and of $H_1$ versus $H_2$. The power function of $T_{01}$ is examined first. With $t > 0$ fixed, we denote this power function by $\pi_{01}(\cdot)$.

If $k = 2$, then the testing situation is the classical one-sided test of $\mu_1 = \mu_2$ versus $\mu_1 < \mu_2$. In this case, rejecting $\mu_1 = \mu_2$ for large values of $T_{01}$ is equivalent to rejecting for large values of $\bar{X}_2 - \bar{X}_1$. It is well known that such a test is UMP and its power function is increasing in $\mu_2 - \mu_1$ (cf. Problem 3.2 on p. 117, Lehmann (1959)). The complexity of the situation increases rapidly in $k$. For $k = 3$ and $w_1 = w_2 = w_3$, Bartholomew (1961) derived an expression for the power function of $T_{01}$. (The derivation is also given in Section 3.4 of Barlow et al. (1972).) Let $\mu \in R^3$, $\Delta = (\Sigma_{i=1}^3(\mu_i - E(\mu \mid H_0))^2)^{1/2}$ and let $\beta$ be defined by

$$(\mu_2 - \mu_1)/\sqrt{2} = \Delta \sin \beta \quad \text{and} \quad (2\mu_3 - \mu_2 - \mu_1)/\sqrt{6} = \Delta \cos \beta.$$

The restriction $\mu \in H_1$ is equivalent to $0 \leq \beta \leq \pi/3$. With $\Phi$ the standard normal distribution function and $\psi(x,t) = (x\Phi(x-\sqrt{t}) + \phi(x-\sqrt{t}))/\phi(x)$,

$$(3.1) \qquad \pi_{01}(\mu) = P_\mu[T_{01} > t] = \frac{\exp(-\frac{1}{2}\Delta^2)}{2\pi} \int_{\pi/6+\beta}^{\pi/2+\beta} \psi(\Delta \sin \theta, t)\,d\theta$$

$$+ \Phi(-\Delta \sin \beta)\Phi(\Delta \cos \beta - \sqrt{t})$$

$$+ \Phi(-\Delta \sin(\pi/3 - \beta))\Phi(\Delta \cos(\pi/3 - \beta) - \sqrt{t}).$$

Bartholomew (1961) took the partial derivative with respect to $\beta$ and noted that for a fixed value of $\Delta$, the power function, which is periodic with period $2\pi$, is increasing for $\beta \in [-5\pi/6, \pi/6]$ and is decreasing for $\beta \in [\pi/6, 7\pi/6]$. Thus, it has a maximum at $\beta = \pi/6$ (the middle of $H_1$), a minimum at $\beta = 7\pi/6$ (the middle of $H_1^*$) and since it is symmetric about $\beta = \pi/6$, it has two equal minima within $H_1$ at $\beta = 0$ and $\beta = \pi/3$.

The partial derivative of (3.1) with respect to $\Delta$, evaluated at $\Delta = 0$, is $\phi(\sqrt{t})(\sqrt{3}/2+\sqrt{t}/(2\pi))\cos(\beta-\pi/6)$, which is positive for $\beta \in (-\pi/3, 2\pi/3)$ and negative for $\beta \in (2\pi/3, 5\pi/3)$. This might lead one to conjecture that the power of $T_{01}$ is increasing in $\Delta \geq 0$ for fixed $\beta \in (-\pi/3, 2\pi/3)$. We have not been able to establish that using (3.1). However, the results given in this section (Corollary 3.2) imply that it is increasing in $\Delta \geq 0$ for fixed $\beta \in (-\pi/6, \pi/2)$. We have also shown that this is the case for $k = 3$ and $\beta \in (-\pi/3, 2\pi/3)$ using techniques similar to those employed in Section 2, but because of their special nature these arguments are not given here. Applying the results of Theorem 3.3 and those concerning the sign of the derivative, at $\Delta = 0$, of the power, we know that for $\beta \in (-\pi/2, -\pi/6) \cup (2\pi/3, 5\pi/6)$ the power function first decreases in $\Delta$ and then approaches 1 as $\Delta \to \infty$.

Bartholomew (1961) also considered the case $k = 4$, but the expression for the power function is quite complicated. Several values of the power function for $k = 4$ are computed there and results like those obtained for $k = 3$ are conjectured to hold for $k = 4$, but no further analysis of the power function is given for $k = 4$.

Remark. If $\alpha \geq -1$ and $\nu \in R^k$ then the distance from $\nu + \alpha E_W(\nu \mid H_1)$ to $H_1$ is the same as the distance from $\nu$ to $H_1$.

Proof. Using B.1 and B.2 on page 131 of Barlow et al. (1972), it can be shown that the level sets for $E_W(\nu + \alpha E(\nu \mid H_1) \mid H_1)$ are the same as the level sets for $E_W(\nu \mid H_1)$. Write out the square of the distance from $\nu + \alpha E_W(\nu \mid H_1)$ to $H_1$ as a sum conditioned on the index being in each level set and the result follows easily.

One interpretation of this Remark is that for any $\nu \notin H_1$ the collection of points $\{\nu + \alpha E_W(\nu \mid H_1);\ \alpha \geq -1\}$ is a set which is parallel to the boundary of $H_1$ (cf. Figure 3.1). Theorem 3.1 gives some properties of the power function of $T_{01}$ as the parameter ranges over such a collection of points.

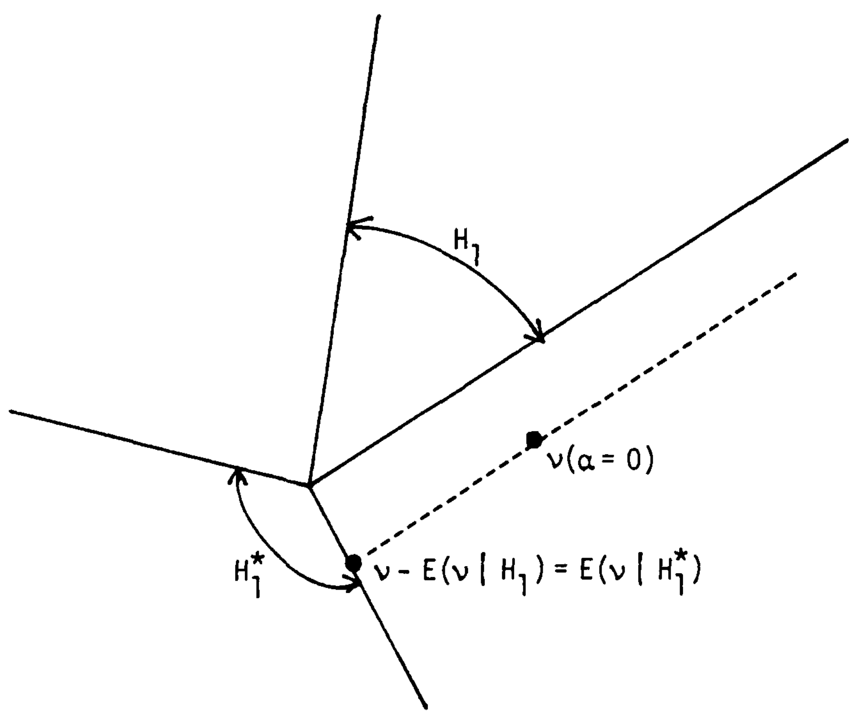



Figure 3.1

<u>Theorem 3.1</u>. Let $\nu \in R^k$. As a function of $\alpha \in (-\infty,\infty)$, $\pi_{01}(\nu + \alpha E_W(\nu \mid H_1))$ is nondecreasing and $\pi_{01}(\nu + \alpha E_W(\nu \mid H_1^{*W}))$ is nonincreasing.

<u>Proof</u>. Write $\nu + \alpha E_W(\nu \mid H_1) = \nu - E_W(\nu \mid H_1) + (\alpha+1)E_W(\nu \mid H_1)$, set $\mu_0 = \nu - E_W(\nu \mid H_1) = E_W(\nu \mid H_1^{*W})$ and $\mu = E_W(\nu \mid H_1)$ and note that by (2.1), $(\mu_0,\mu)_W = 0$. Set $S = H_0^{\perp W} = \{x \in R^k : \sum_{i=1}^k w_i x_i = 0\}$ and $C = C_{01} = H_1 \cap S$. Since $S^{\perp W} = H_0$, $C_{01} \oplus S^{\perp W} = H_1$. Applying Theorem 2.10, $P_{\delta\mu,W}(A_W^C) = P_{\mu_0+\delta\mu,W}[T_{01} > t]$ is nondecreasing in $\delta$. Thus, $\pi_{01}(\mu_0+\delta\mu)$ is nondecreasing in $\delta \geq 0$. For $\delta \leq 0$, consider $\pi_{01}(\mu_0+(-\delta)(-\mu))$, which is nonincreasing in $-\delta$ if $-\mu \in C_{01}^{*W} = (H_1 \cap H_0^{\perp W})^{*W} = H_1^{*W} \oplus H_0$. Barlow et al. (1972, p. 49) show that

$$(3.2) \qquad H_1^{*W} = \{x \in R^k : \textstyle\sum_{j=1}^i w_j x_j \geq 0 \text{ for } i = 1,2,\cdots,k-1 \text{ and } \sum_{j=1}^k w_j x_j = 0\}.$$

Since $(-H_1) \cap H_0^{\perp W} \subset H_1^{*W}$, $-\mu + (\sum_{j=1}^k w_j \mu_j)e_k \in H_1^{*W}$ and $-\mu \in H_1^{*W} \oplus H_0$, the first claim is established.

For the second conclusion, write $\nu + \alpha E_W(\nu \mid H_1^{*W})$ as $\mu_0 + (\alpha+1)\mu$ with $\mu_0 = E_W(\nu \mid H_1)$ and $\mu = E_W(\nu \mid H_1^{*W})$ and note that $(\mu_0,\mu)_W = 0$. Let $S$ and $C$ be as in the first part of the proof. Now $\mu \in H_1^{*W} \subset C_{01}^{*W} = (H_1 \cap S)^{*W} = H_1^{*W} \oplus H_0$, and applying Theorem 2.10, $\pi_{01}(\mu_0+\delta\mu)$ is nonincreasing in $\delta \geq 0$. For $\delta \leq 0$, consider $\pi_{01}(\mu_0+(-\delta)(-\mu))$ and note that $-\mu \in -H_1^{*W} \subset -C^{*W}$. Thus, applying Theorem 2.10 again, $\pi_{01}(\mu_0+\delta\mu)$ is nonincreasing for $\delta \leq 0$. The proof is completed.

Theorem 3.1 can also be established using the results in Robertson and Wright (1982). They considered two relations on $R^k$ defined by $x \lesssim y$

provided $y-x \in H_1$ and $x \ll y$ provided $y-x \in -H_1^{*W}$ and proved that if either $\mu \lesssim \mu'$ or $\mu \ll \mu'$ then $\pi_{01}(\mu) \le \pi_{01}(\mu')$ (i.e., $\pi_{01}(\cdot)$ is isotone with respect to both $\lesssim$ and $\ll$). For $\alpha_1 \le \alpha_2$, $\nu + \alpha_1 E_W(\nu \mid H_1) \lesssim \nu + \alpha_2 E_W(\nu \mid H_1)$ and $\nu + \alpha_2 E_W(\nu \mid H_1^{*W}) \ll \nu + \alpha_1 E_W(\nu \mid H_1^{*W})$ and the results of Theorem 3.2 follow immediately. However, this approach does not yield the analogous results for $T_{12}$.

Corollary 3.2. If $\mu \in -H_1^{*W} \oplus H_0$, then $\pi_{01}(\delta\mu)$ is nondecreasing in $\delta \in (-\infty,\infty)$. Furthermore, $T_{01}$ is unbiased.

Proof. By hypothesis, $\mu = \nu + \nu'$ with $\nu \in -H_1^{*W}$ and $\nu' \in H_0$. Examining the definition of $T_{01}$, it is clear that $\pi_{01}(\delta\mu) = \pi_{01}(\delta\nu)$. Applying the second conclusion of Theorem 3.1 with $\nu$ replaced by $-\nu$, we see that $\pi_{01}(\delta\nu)$ is nondecreasing in $\delta \in (-\infty,\infty)$.

In the proof of Theorem 3.1, we saw that $H_1 \subset -H_1^{*W} \oplus H_0$. So for $\mu \in H_1 - H_0$, $\pi_{01}(\mu) \ge \pi_{01}(0 \cdot \mu)$. The proof is completed.

Remark. Since $H_1 \subset H_1^{*W} \oplus H_0$, $\pi_{01}(\delta\mu)$ is nondecreasing in $\delta \in (-\infty,\infty)$ if $\mu \in H_1$.

The limiting behavior of $\pi_{01}$, in the directions considered in Theorem 3.1 and Corollary 3.2, will be discussed next. A slight generalization of Theorem 3.4 of Barlow et al. (1972) will be used to obtain the limits of interest. Following their notation, we define for $\mu \in R^k$,

$$\Delta^2 = \Delta^2(\mu) = \Sigma_{i=1}^k w_i(\mu_i - \tilde{\mu})^2 \text{ with } \tilde{\mu} = \Sigma_{i=1}^k w_i\mu_i / \Sigma_{i=1}^k w_i.$$

We first note that the statement of the theorem given there needs some clarification. Consider the following example: Let $w = e_k$, let $\nu \in H_1$

with $\|\nu-\tilde{\nu}\| > 0$ and let $\mu_n = (-1)^n n\nu$. Then $\Delta^2(\mu_n) = n^2\Delta^2(\nu) \to \infty$ and $(\mu_{n,i}-\tilde{\mu}_n)/(\mu_{n,j}-\tilde{\mu}_n) = (\nu_i-\tilde{\nu})/(\nu_j-\tilde{\nu})$ for $1 \le i, j \le k$. However, for odd $n$ the number of distinct values in $E(\mu_n \mid H_1)$ is 1, i.e., $E(\mu_n \mid H_1)$ is constant, but for $n$ even, $E(\mu_n \mid H_1) = n \cdot \nu$ which is not constant since $\|\nu-\tilde{\nu}\| > 0$. Hence, the hypotheses of their Theorem 3.4 may hold, but $\ell_\mu$ may not be constant. By making only slight modifications of their proof, one can prove the following generalization.

Theorem 3.3. Let $v, \theta, \eta_n \in R^k$ with $\eta_n \to 0$ as $n \to \infty$. If $w_n = a_n v$ and $\mu_n - \tilde{\mu}_n = b_n^{1/2}(\theta - \tilde{\theta} + \eta_n)$ with $a_n b_n \to \infty$ as $n \to \infty$, then $\pi_{01}(\mu_n) \to 1$ provided $\theta \notin H_1^{*W} \oplus H_0$, and $\pi_{01}(\mu_n) \to 0$ if $\theta \in (H_1^{*W} \oplus H_0)^0$, where $A^0$ denotes the interior of $A$.

Barlow et al. (1972) applied this result to show that $T_{01}$ is consistent for $\mu \notin H_1^{*W} \oplus H_0$. It also gives the radial limits of the power function in certain directions. These limits are obtained by setting $\nu = 0$ in the following:

Corollary 3.4. Let $\mu, \nu \in R^k$. If $\mu \notin H_1^{*W} \oplus H_0$, then $\lim_{\delta\to\infty} \pi_{01}(\nu+\delta\mu) = 1$. If $\mu \in (H_1^{*W} \oplus H_0)^0$, then $\lim_{\delta\to\infty} \pi_{01}(\nu+\delta\mu) = 0$.

Proof. The result follows from Theorem 3.3 by setting $v = w$, $\theta = \mu$, $\eta_n = (\nu-\tilde{\nu})/\delta_n$, $a_n \equiv 1$, $b_n = \delta_n^2$ with $\delta_n \to \infty$.

One can obtain the form of $\lim_{\delta\to\infty} \pi_{01}(\nu+\delta\mu)$ with $\mu \in \partial(H_1^{*W} \oplus H_0)$, but such limits play no central role in this work and are tedious to develop, so they are not included. Next, we study the limits of the power function along lines parallel to $H_1$ and $H_1^{*W}$, that is, we consider $\lim_{\alpha\to\pm\infty} \pi_{01}(\nu+\alpha E_W(\nu \mid H_1))$ and $\lim_{\alpha\to\pm\infty} \pi_{01}(\nu+\alpha E_W(\nu \mid H_1^{*W}))$. Directions parallel to $H_1$ will be discussed first.

Corollary 3.5. Let $\nu \in R^k$. If $\nu \in H_1^{*W} \oplus H_0$, then $\pi_{01}(\nu+\alpha E_W(\nu \mid H_1)) = \pi_{01}(\nu)$. If $\nu \notin H_1^{*W} \oplus H_0$, then $\lim_{\alpha\to\infty} \pi_{01}(\nu+\alpha E_W(\nu \mid H)) = 1$ and $\lim_{\alpha\to-\infty} \pi_{01}(\nu+\alpha E_W(\nu \mid H_1)) = 0$.

Proof. Using (3.2), we characterize $H_1^{*W} \oplus H_0$ as follows:

$$(3.3) \qquad H_1^{*W} \oplus H_0 = \{x \in R^k : \Sigma_{j=1}^{i} w_j x_j \geq (\Sigma_{j=1}^{i} w_j)\tilde{x} \text{ for } i = 1,2,\cdots,k-1\}.$$

Using the minimum lower sets algorithm for computing $E(x \mid H_1)$ (cf. Barlow et al. (1972, p. 76)), we see that $E_W(x \mid H_1) \in H_0$ if and only if $x \in H_1^{*W} \oplus H_0$. If $\nu \in H_1^{*W} \oplus H_0$, then the first conclusion follows from the fact that $\pi_{01}$ is invariant under constant shifts. If $\nu \notin H_1^{*W} \oplus H_0$, then $E_W(\nu \mid H_1) \notin H_0$ and since $H_1 \cap (H_1^{*W} \oplus H_0) = H_0$, $E_W(\nu \mid H_1) \notin H_1^{*W} \oplus H_0$. Applying Corollary 3.4 gives the second conclusion.

For the last conclusion, we consider $\pi_{01}(\nu+(-\alpha)(-E_W(\nu \mid H_1)))$. The desired result follows from Corollary 3.4 by showing that $-E_W(\nu \mid H_1)$ is in the interior of $H_1^{*W} \oplus H_0$, which is characterized by making the inequalities in (3.3) strict. Now $-E_W(\nu \mid H_1) \notin H_0$ and has nonincreasing coordinates, and for any $x \in R^k$ with these two properties, $\Sigma_{j=1}^{i} w_j x_j/\Sigma_{j=1}^{i} w_j$ is nonincreasing in $i$ and equals $\tilde{x}$ for $i = k$. Furthermore, $\Sigma_{j=1}^{k-1} w_j x_j/\Sigma_{j=1}^{k-1} w_j > \tilde{x}$, for if not $x$ is constant. The proof is completed.

We now consider limits along lines parallel to $H_1^{*W}$. If $\nu \in H_1^{*W}$, then $\nu+\alpha E_W(\nu \mid H_1^{*W}) = (\alpha+1)\nu$ and yields radial limits as $\alpha \to \infty$. If $\nu \notin H_1^{*W}$, then $E_W(\nu \mid H_1^{*W}) \in \partial(H_1^{*W} \oplus H_0)$ and $\lim_{\alpha\to\infty} \pi_{01}(\nu+\alpha E_W(\nu \mid H_1^{*W}))$ is the type of limit discussed after the proof of Corollary 3.4.

Corollary 3.6. Let $\nu \in R^k$. If $\nu \in H_1$, then $\pi_{01}(\nu+\alpha E_W(\nu \mid H_1^{*W})) = \pi_{01}(\nu)$. If $\nu \notin H_1$, then $\lim_{\alpha\to-\infty} \pi_{01}(\nu+\alpha E_W(\nu \mid H_1^{*W})) = 1$.

Proof. Since $E_W(\nu \mid H_1^{*W}) = \nu - E_W(\nu \mid H_1)$, we see that $E_W(\nu \mid H_1^{*W}) = 0$ if and only if $\nu \in H_1$. The first conclusion is now clear.

For the second part, assume $\nu \notin H_1$. Using (3.2) and (3.3), we see that $x \in H_1^{*W} \oplus H_0 \Longleftrightarrow x-\tilde{x} \in H_1^{*W}$. The proof is completed by applying Corollary 3.4 provided $-E_W(\nu \mid H_1^{*W}) \notin H_1^{*W} \oplus H_0$. If $-E_W(\nu \mid H_1^{*W}) \in H_1^{*W} \oplus H_0$, then $-E_W(\nu \mid H_1^{*W}) \in H_1^{*W}$. But $(-H_1^{*W}) \cap H_1^{*W} = \{0\}$, and if $E_W(\nu \mid H_1^{*W}) = 0$, then $\nu \in H_1$.

We now turn our attention to the study of $\pi_{12}$, the power function of $T_{12}$. For $k = 2$ rejecting $\mu_1 \le \mu_2$ in favor of $\mu_1 > \mu_2$ for large values of $T_{12}$ is equivalent to rejecting if $\overline{X}_1-\overline{X}_2$ is large. This test is known to be unbiased, UMP and to have a power function which is nondecreasing in $\mu_1-\mu_2$. For $k = 3$ and $w_1 = w_2 = w_3$, one can employ the same techniques used by Bartholomew (1961) to show that

$$(3.4) \qquad \pi_{12}(\mu) = P_\mu[T_{12} > t] = \frac{\exp(-\frac{1}{2}\Delta^2)}{2\pi} \int_{\beta-\pi}^{\beta-\pi/3} \psi(\Delta \sin\theta, t)\,d\theta + \Phi(-\Delta \sin\beta - \sqrt{t})\Phi(\Delta\cos\beta) + \Phi(-\Delta\cos(\beta+\pi/6)-\sqrt{t})\Phi(\Delta\sin(\beta+\pi/6)),$$

with $\Delta$ and $\beta$ defined as before. It is not difficult to show that, for fixed $\Delta$, $\pi_{12}(\cdot)$ is antisymmetric about $\beta = \pi/6$ and $\beta = 7\pi/6$. Based on Bartholomew's work on $\pi_{01}$ with $k = 3$ and $w_1 = w_2 = w_3$, one would conjecture that $\pi_{12}$ is, for fixed $\Delta$, decreasing for $\beta \in [-5\pi/6, \pi/6]$ and increasing for $\beta \in [\pi/6, 7\pi/6]$. We have not established this analytically, but have numerically obtained the value of $\partial\pi_{12}/\partial\beta$ for several values of $\beta$, $\Delta$ and $t$. This partial derivative appears to be negative on $(-5\pi/6, \pi/6)$ and positive on $(\pi/6, 7\pi/6)$.

The partial derivative of $\pi_{12}$, with respect to $\Delta$, evaluated at $\Delta = 0$ is $-\phi(\sqrt{t})(3t/(2\pi))^{1/2} + 1/2)\cos(\beta-\pi/6)$, which is negative for $\beta \in (-\pi/3, 2\pi/3)$ and positive for $\beta \in (2\pi/3, 5\pi/3)$. As one might expect, this behavior is opposite to that of $\pi_{01}$.

For arbitrary $k$, we apply the results in Section 2 in our study of $\pi_{12}$.

<u>Theorem 3.7</u>. Let $\nu \in R^k$. As a function of $\alpha$, $\pi_{12}(\nu + \alpha E_W(\nu \mid H_1))$ is nonincreasing for $-\infty < \alpha < \infty$ and $\pi_{12}(\nu + \alpha E_W(\nu \mid H_1^{*W}))$ is nondecreasing for $\alpha \geq -1$.

<u>Proof</u>. Write $\nu + \alpha E_W(\nu \mid H_1) = E_W(\nu \mid H_1^{*W}) + (\alpha+1)E_W(\nu \mid H_1) = \mu_0 + (\alpha+1)\mu$ with $(\mu_0,\mu)_W = 0$. Set $S = H_0^{\perp W} = \{x \in R^k : \Sigma_{i=1}^k w_i x_i = 0\}$ and $C = C_{12} = H_1^{*W} \subset S$. Applying Theorem 2.10, we see that for $\mu \in H_1 = C_{12}^{*W}$, $P_{\delta\mu,W}(A_W^C) = \pi_{12}(\mu_0 + \delta\mu)$ is nonincreasing in $\delta$ for $\delta \geq 0$. For $\delta \leq 0$, consider $\pi_{12}(\mu_0 + (-\delta)(-\mu))$ which is nondecreasing in $-\delta$ since $-\mu \in -H_1 = -C_{12}^{*W}$.

For the second conclusion, $\nu + \alpha E_W(\nu \mid H_1^{*W}) = E_W(\nu \mid H_1) + (\alpha+1)E_W(\nu \mid H_1^{*W}) = \mu_0 + (\alpha+1)\mu$ with $(\mu_0,\mu)_W = 0$. Since $\mu \in H_1^{*W} = C_{12}$, we apply Theorem 2.10 to show that $\pi_{12}(\mu_0 + \delta\mu)$ is nondecreasing in $\delta \geq 0$. The proof of the Theorem is completed.

Comparing Theorems 3.1 and 3.7, we see that $\pi_{01}(\nu + \alpha E_W(\nu \mid H_1^{*W}))$ is monotone for $-\infty < \alpha < \infty$, but $\pi_{12}(\nu + \alpha E_W(\nu \mid H_1^{*W}))$ is only claimed to be monotone for $\alpha \geq -1$. We consider an example to show that the second conclusion of Theorem 3.7 is not valid for $-\infty < \alpha < \infty$.

Example. Let $k = 3$, $w = e_3$, $\nu = (\sqrt{2}/2,-\sqrt{2}/2,0) \in H_1^*$. (Recall, $H_1^*$ is characterized in (3.2).) Now $\nu + \alpha E(\nu \mid H_1^*) = (\alpha+1)\nu$, and we will show that $\pi_{12}((\alpha+1)\nu)$ is not monotone in $(-\infty,-1)$. If it were, then $\pi_{12}(\delta(-\nu))$ would be monotone for $\delta > 0$. However, the $\beta$ corresponding to $-\nu$ is $\pi/2$ and for such $\beta$, $\partial\pi_{12}/\partial\Delta|_{\Delta=0} < 0$. Hence, the power decreases for $\delta$ small and positive, but applying Corollary 3.11, we see that $\lim_{\delta\to\infty} \pi_{12}(\delta(-\nu)) = 1$.

The above example is interesting for several other reasons, also.

(1) For the $\nu$ chosen, $\nu + \alpha E_W(\nu \mid H_1^*) = (\alpha+1)\nu$ and so we see that $\pi_{12}(\delta\nu)$ is not monotone for $\delta < 0$ (see the next corollary).

(2) It shows that $\pi_{12}$ is not antitone with respect to the partial order, $\ll$, discussed in Robertson and Wright (1982), that is, if $0 \leq \delta < \delta'$, then $\delta(-\nu) \ll \delta'(-\nu)$ but $\pi_{12}(-\delta\nu)$ may be less than $\pi_{12}(-\delta'\nu)$.

(3) It shows that $T_{12}$ is biased. Along the ray $\{\delta(-\nu) : \delta \geq 0\}$, the power decreases for small, positive $\delta$ and so the level of significance is at least (and, in fact, is equal to) $\pi_{12}(0) = \pi_{12}(0(-\nu)) > \pi_{12}(\delta(-\nu))$ for some $\delta > 0$. We will consider the question of the unbiasedness of $T_{12}$ in more detail later in this section.

Corollary 3.8. *If* $\mu \in H_1$, *then* $\pi_{12}(\delta\mu)$ *is nonincreasing for* $\delta \in (-\infty,\infty)$. *If* $\mu \in H_1^{*W} \oplus H_0$, *then* $\pi_{12}(\delta\mu)$ *is nondecreasing for* $\delta \in [0,\infty)$.

Proof. Corollary 3.8 follows from Theorem 3.7 just as Corollary 3.2

follows from Theorem 3.1.

To obtain the limits of $\pi_{12}$ in the directions considered in Theorem 3.7 and Corollary 3.8, we establish the analogue of Theorem 3.3 for $\pi_{12}$.

Theorem 3.9. Let $v,\theta,\eta_n \in R^k$ with $\eta_n \to 0$ as $n \to \infty$. If $w_n = a_n v$ and $\mu_n - \tilde{\mu}_n = b_n^{1/2}(\theta - \tilde{\theta} + \eta_n)$ with $a_n b_n \to \infty$ as $n \to \infty$, then $\pi_{12}(\mu_n) \to 1$ provided $\theta \notin H_1$ and $\pi_{12}(\mu_n) \to 0$ if $\theta \in H_1^0$.

Proof. The proof of the first conclusion is very similar to the proof given in Barlow et al. (1972) for their Theorem 3.4. The LRT of $H_1$ versus $H_2$ rejects $H_1$ if $T_{12} > t$. In this case, $T_{12}/(a_n b_n)$ converges in probability to $\sum_{i=1}^k v_i(E(\theta \mid H_1) - \theta\ )^2$, which is positive if $\theta \notin H_1$. Of course, $t/(a_n b_n) \to 0$ and the first conclusion is established.

For the second conclusion, $\pi_{12}(\mu_n) = P_{\mu_n,W}\{x : \|E_W(x \mid H_1) - x\|_W^2 > t\}$. But $E_W(\cdot \mid H_1) = E_V(\cdot \mid H_1)$ where $V$ is a kxk diagonal matrix with $V_{ii} = v_i$ for $i = 1,2,\cdots,k$, $\|E_V(x \mid H_1) - x\|_W^2 = \|E_V(a_n^{1/2}x \mid H_1) - a_n^{1/2}x\|_V^2$, and if $x \sim \eta(\mu_n, W^{-1})$ then $a_n^{1/2}x \sim \eta(a_n^{1/2}\mu_n, V^{-1})$. So, $\pi_{12}(\mu_n)$ $= P_{0,V}\{x : \|E_V(x + a_n^{1/2}\mu_n \mid H_1) - x - a_n^{1/2}\mu_n\|_V^2 > t\}$. By the hypotheses of the theorem, $a_n^{1/2}\mu_n = (a_n b_n)^{1/2}(\theta - \tilde{\theta} + \eta_n)$, $a_n b_n \to \infty$, $\eta_n \to 0$ and $\theta - \tilde{\theta}$ has strictly increasing coordinates. Thus, for each $x \in R^k$, there exists an $n(x)$, with $x + a_n^{1/2}\mu_n \in H_1$ for all $n \geq n(x)$. Hence,

$$\|E_V(x + a_n^{1/2}\mu_n \mid H_1) - x - a_n^{1/2}\mu_n\|_V^2 = 0 \quad \text{for all} \quad n \geq n(x),$$

and because $t > 0$, the desired result is established.

Corollary 3.10. If $n_i = n\gamma_i$ with $\gamma_i > 0$ for $i = 1,2,\cdots,k$, then $T_{12}$ is consistent for all $\mu \notin H_1$.

Proof. This result follows from Theorem 3.9 by setting $v = (\gamma_1/\sigma_1^2, \cdots, \gamma_k/\sigma_k^2)$, $\theta = \mu$, $\eta_n \equiv 0$, $b_n \equiv 1$, and $a_n = n$.

Corollary 3.11. Let $\mu, \nu \in R^k$. If $\mu \notin H_1$, then $\lim_{\delta\to\infty} \pi_{12}(\nu+\delta\mu) = 1$, and if $\mu \in H_1^0$, then $\lim_{\delta\to\infty} \pi_{12}(\nu+\delta\mu) = 0$.

Corollary 3.11 follows immediately from Theorem 3.9 and with $\nu = 0$, gives the values of certain radial limits. Because the radial limits of $\pi_{12}$ for $\mu \in \partial H_1$ are of interest in our study of the bias of $T_{12}$, we need to obtain the value of these limits. For $\mu \in H_1$, let $1 \le j_1 < j_2 < \cdots < j_h = k$ be defined by $\mu_1 = \cdots = \mu_{j_1} < \mu_{j_1+1} = \cdots = \mu_{j_2} < \cdots < \mu_{j_{h-1}+1} = \cdots = \mu_{j_h}$, set

$$(3.5) \qquad C'(\mu) = \{x \in R^k : x_1 \le x_2 \le \cdots \le x_{j_1}, x_{j_1+1} \le \cdots \le x_{j_2}, \cdots, x_{j_{h-1}+1} \le \cdots \le x_{j_h}\}$$

and set $G_1 = \{1,2,\cdots,j_1\}$, $G_2 = \{j_1+1,\cdots,j_2\},\cdots,G_h = \{j_{h-1}+1,\cdots,j_h\}$. The $G_\ell$ are the level sets of $\mu$.

Theorem 3.12. Let $\nu \in R^k$ and $\mu \in H_1$. Then,

$$(3.6) \qquad \lim_{\delta\to\infty} \pi_{12}(\nu+\delta\mu) = P_{\nu,W}[\|E_W(x \mid C'(\mu))-x\|_W^2 > t].$$

Proof. If $\mu \in H_1^0$, then $C'(\mu) = R^k$, the r.h.s. of (3.6) is zero and (3.6) follows from Corollary 3.11. Suppose $\mu \in \partial H_1$ and consider $E_W(x+\nu+\delta\mu \mid H_1)$. For a fixed $x$, $x_i + \nu_i + \delta\mu_i - x_j - \nu_j - \delta\mu_j \to \infty$ as $\delta \to \infty$ for $i \in G_\ell$, $j \in G_{\ell'}$ with $\ell' < \ell$. So for each fixed $x$, there exists a $\delta(x)$ with

$$\max_{i\in G_1}(x_i+\nu_i+\delta\mu_i) < \min_{i\in G_2}(x_i+\nu_i+\delta\mu_i) \le \max_{i\in G_2}(x_i+\nu_i+\delta\mu_i)$$
$$< \cdots < \min_{i\in G_h}(x_i+\nu_i+\delta\mu_i)$$

for $\delta \geq \delta(x)$. It follows from the minimum lower sets algorithm (cf. Barlow et al. (1972, p. 76)) that for $i_0 \in G_\ell$, $\min_{i\in G_\ell} y_i \le y_{i_0} \le \max_{i\in G_\ell} y_i$ and that since $\delta\mu$ is constant on the level sets of $\mu$, $E_W(y+\delta\mu)C'(\mu)) = E_W(y)C'(\mu)) + \delta\mu$. Since $C'(\mu) \supset H_1$, $E_W(x+\nu+\delta\mu \mid C'(\mu)) = E_W(x+\nu+\delta\mu \mid H_1)$ for $\delta \geq \delta(x)$. Hence, for each $x$, $\|E_W(x+\nu+\delta\mu \mid H_1) - x - \nu - \delta\mu\|_W \to \|E_W(x \mid C'(\mu)) - x\|_W$. Thus,

$$\pi_{12}(\nu+\delta\mu) = P_{0,W}\{x : \|E_W(x+\nu+\delta\mu \mid H_1) - x - \nu - \delta\mu\|_W^2 > t\} \to$$
$$P_{0,W}\{x : \|E_W(x+\nu \mid C'(\mu)) - x - \nu\|_W^2 > t\}$$
$$= P_{\nu,W}[\|E_W(x \mid C'(\mu)) - x\|_W^2 > t].$$

The proof is completed.

By taking $\nu = 0$ in Theorem 3.12, we obtain the radial limits for $\mu \in \partial H_1$. Corollary 2.6 of Robertson and Wegman shows that the r.h.s. of (3.6) with $\nu = 0$ is a weighted sum of $\chi^2$ tail probabilities, and the remark on p. 148 of Barlow et al. (1972) shows that the weighting constants, i.e., level probabilities, in this case, are convolutions of those for a total order. We will compute some values for this limit when we study bias. However, since $C'(\mu) \supset H_1$, $\|E_W(x \mid H_1) - x\|_W \ge \|E_W(x \mid C'(\mu)) - x\|_W$ and so, $P_{\nu,W}[\|E_W(x \mid C'(\mu)) - x\|_W > t] \le \pi_{12}(\nu)$.

Next, we study the limits of the power function along lines parallel to $H_1$ and $H_1^{*W}$. Directions parallel to $H_1$ will be discussed first. If $\nu \in H_1$, then $\nu + \alpha E_W(\nu \mid H_1) = (\alpha+1)\nu$, which yields a radial limit as $\alpha \to \infty$. So we may suppose $\nu \notin H_1$.

Corollary 3.13. If $\nu \notin H_1$, then

$$\lim_{\alpha\to\infty} \pi_{12}(\nu + \alpha E_W(\nu \mid H_1)) = P_{\nu,W}[\|E_W(x \mid C'(E_W(\nu \mid H_1)))-x\|_W > t].$$

If $\nu \in H_1^{*W} \oplus H_0$, then $\pi_{12}(\nu+\alpha E_W(\nu \mid H_1)) = \pi_{12}(\nu)$, and if $\nu \notin H_1^{*W} \oplus H_0$, then

$$\lim_{\alpha\to-\infty} \pi_{12}(\nu+\alpha E_W(\nu \mid H_1)) = 1.$$

Proof. The first part of the result follows from Theorem 3.12. For the second part, we recall that $E_W(\nu \mid H_1) \in H_0$ if and only if $\nu \in H_1^{*W} \oplus H_0$. So, if $\nu \in H_1^{*W} \oplus H_0$, then $\pi_{12}(\nu+\alpha E_W(\nu \mid H_1)) = \pi_{12}(\nu)$. If $\nu \notin H_1^{*W} \oplus H_0$, then $-E_W(\nu \mid H_1) \notin H_1$ because $(-H_1) \cap H_1 = H_0$. Applying Corollary 3.11 gives the desired conclusion, and the proof is completed.

Directions parallel to $H_1^{*W}$ are considered next. We may assume $\nu \notin H_1^{*W}$, and in fact, since $\pi_{12}$ is invariant under constant shifts, we may assume $\nu \notin H_1^{*W} \oplus H_0$.

Corollary 3.14. If $\nu \in H_1$, then $\pi_{12}(\nu+\alpha E_W(\nu \mid H_1^{*W})) = \pi_{12}(\nu)$. If $\nu \notin H_1$, then $\lim_{\alpha\to\infty} \pi_{12}(\nu+\alpha E_W(\nu \mid H_1^{*W})) = 1$. If $\nu \notin H_1 \cup (H_1^{*W} \oplus H_0)$, then $\lim_{\alpha\to-\infty} \pi_{12}(\nu+\alpha E_W(\nu \mid H_1^{*W})) = 1$

Proof. If $\nu \in H_1$, then $E_W(\nu \mid H_1^{*W}) = 0$ and so the first conclusion is immediate. If $\nu \notin H_1$, or equivalently $E_W(\nu \mid H_1^{*W}) \notin H_1$, then appealing to Corollary 3.11, $\lim_{\alpha\to\infty} \pi_{12}(\nu+\alpha E_W(\nu \mid H_1^{*W}) = 1$. The last conclusion also follows from Corollary 3.11, if we can show that $-E_W(\nu \mid H_1^{*W}) \notin H_1$. Suppose $\eta = -E_W(\nu \mid H_1^{*W}) \in H_1$. Then $\Sigma_{j=1}^{k} w_j\eta_j = 0$, $\eta_j$ is nondecreasing and $\eta \notin H_0$ $(E_W(\nu \mid H_1^{*W}) \in H_0 \Leftrightarrow E_W(\nu \mid H_1^{*W}) = 0 \Leftrightarrow \nu \in H_1)$. So, $\Sigma_{j=1}^{i} w_j\eta_j$ is nondecreasing in $i$ and equal to zero for $i = k$. Since $\eta_k \neq 0$, $\Sigma_{j=1}^{i} w_j\eta_j < 0$ for $i = 1,2,\cdots,k-1$. If $G_1 = \{1,2,\cdots,j_1\}$ is the first level set for $E_W(\nu \mid H_1)$, then $j_1 < k$ and $\Sigma_{j=1}^{j_1} w_j\eta_j$ $= \Sigma_{j=1}^{j_1} w_j(E_W(\nu \mid H_1)_j - \nu_j) = \Sigma_{j=1}^{j_1} w_j(\Sigma_{\ell=1}^{j_1} w_\ell\nu_\ell/\Sigma_{\ell=1}^{j_1} w_\ell) - \Sigma_{j=1}^{j_1} w_j\nu_j = 0$. This contradiction completes the proof.

We have already noted that $T_{12}$ is biased and we now wish to examine the amount of bias. In the case $k = 3$ with $w = e_3$, the level of significance is $\pi_{12}(0) = P[\chi_2^2 > t]/3 + P[\chi_1^2 > t]/2$. Partition $R^3$ into four sets depending on the number of level sets in the projection onto $H_1$. Specifically, with $x^* = E_W(x \mid H_1)$, let

$$C_1 = \{x : x_1^* < x_2^* < x_3^*\} \ (= H_1^0), \qquad C_2 = \{x : x_1^* = x_2^* < x_3^*\}$$

$$C_3 = \{x : x_1^* < x_2^* = x_3^*\} \quad \text{and} \quad C_4 = \{x : x_1^* = x_2^* = x_3^*\} \ (= H_1^* \oplus H_0).$$

We have seen that $\inf_{\mu\in C_1} \pi_{12}(\mu) = 0$ (cf. Corollary 3.11) and that $\inf_{\mu\in C_4} \pi_{12}(\mu) = \pi_{12}(0)$ (cf. Corollary 3.8). It will be shown (cf. Theorem 3.15) that $\inf_{\mu\in C_2} \pi_{12}(\mu) = \inf_{\mu\in C_3} \pi_{12}(\mu) = P[\chi_1^2 > t]/2$, and so by the continuity of $\pi_{12}$, $\inf_{\mu\notin H_1} \pi_{12}(\mu) = 2^{-1}P[\chi_1^2 > t]$. In the case being considered, the 5% critical value for $T_{12}$ is 4.578 and $P[\chi_1^2 \geq 4.578]/2$ $= .0162$, which gives some idea of the amount of bias. (Larger $k$ will be discussed later.)

Returning to the case of arbitrary $k$, we partition $R^k$ into $M = 2^{k-1}$ subsets depending on the level sets of $x^*$. Let $C_1 = \{x : x_1^* < x_2^* < \cdots < x_k^*\}$ $(= H_1^0)$, $C_2 = \{x : x_1^* = x_2^* < x_3^* < \cdots < x_k^*\}$, $C_3 = \{x : x_1^* < x_2^* = x_3^* < x_4^* < \cdots < x_k^*\}$ $\cdots$, $C_M = \{x : x_1^* = \cdots = x_k^*\}$ $(= H_1^{*W} \oplus H_0)$. For $x \in C_i$, let $C'(x^*)$ be defined as in (3.5) and note that $C'(x^*)$ is the same cone for each $x \in C_i$. Set $C_i' = C'(x^*)$ for $x \in C_i$.

Theorem 3.15. With $C_i$ and $C_i'$ defined as above,

$$(3.7) \qquad \inf_{\mu \in C_i} \pi_{12}(\mu) = P_{0,W}[\|E_W(x \mid C_i') - x\|_W > t].$$

Proof. We consider $C_M = H_1^{*W} \oplus H_0$ first and note that $C_M' = H_1$. Equation (3.7) follows from Corollary 3.8. Fix $i < M$ and let $\mu \in C_i$. Now, as in the proof of the Remark preceding Theorem 3.1, $\mu + \alpha E_W(\mu \mid H_1)$ has the same level sets as $\mu$ for $\alpha \geq -1$. Applying Theorem 3.7, $\pi_{12}(\mu + \alpha E_W(\mu \mid H_1)) \leq \pi_{12}(\mu)$ for $\alpha \geq 0$ so that

$$\inf_{\mu \in C_i} \pi_{12}(\mu) = \inf_{\mu \in C_i} \lim_{\alpha \to \infty} \pi_{12}(\mu + \alpha E_W(\mu \mid H_1))$$
$$= \inf_{\mu \in C_i} P_{\mu,W}[\|E_W(x \mid C_i') - x\|_W^2 > t]$$

by Corollary 3.13.

Let $\mu_\ell = (\mu_{j_{\ell-1}+1}, \cdots, \mu_{j_\ell})$ and $W_\ell$ be the $(j_\ell - j_{\ell-1}) \times (j_\ell - j_{\ell-1})$ diagonal matrix with diagonal elements $w_{j_{\ell-1}+1}, \cdots, w_{j_\ell}$ for $\ell = 1, 2, \cdots, h$. Now, $\|E_W(x \mid C_i') - x\|_W^2 = \sum_{\ell=1}^h \|E_{W_\ell}(x_\ell \mid H_{1,\ell}) - x_\ell\|_{W_\ell}^2$, which is a sum if independent random variables on $R^k$. The distribution of the $\ell^{th}$ summand could be thought of as indexed by $(\mu, W)$ or $(\mu_\ell, W_\ell)$. In the latter case, we can apply (3.7) with $i = M$, to see that

$P_{\mu_\ell,W_\ell}[\|E_{W_\ell}(x_\ell \mid H_{1,\ell})-x_\ell\|^2_{W_\ell} > t] \geq P_{0,W_\ell}[\|E_{W_\ell}(x_\ell \mid H_{1,\ell})-x_\ell\|^2_{W_\ell} > t]$ since $E_W(\mu_\ell \mid H_{1,\ell})$ is constant. Under both probabilities $P_{\mu,W}$ and $P_{0,W}$, $\|E_W(x \mid C_i')-x\|^2_W$ is a sum of $h$ independent random variables with the $\ell^{\underline{th}}$ summand stochastically larger under $P_{\mu,W}$ then under $P_{0,W}$, $\ell = 1,2,\cdots,h$. So $\|E_W(x \mid C_i')-x\|^2_W$ is stochastically larger under $P_{\mu,W}$ (cf. Proposition C.1, p. 485, Marshall and Olkin (1979)). Hence,

$$\inf_{\mu\in C_i} P_{\mu,W}[\|E_W(x \mid C_i')-x\|^2_W > t] \geq P_{0,W}[\|E_W(x \mid C_i')-x\|^2_W > t],$$

and the reverse inequality follows from the fact that $C_i$ is a cone and $P_{\mu,W}$ is continuous in $\mu$.

<u>Corollary 3.16</u>. $\inf_{\mu\notin H_1} \pi_{12}(\mu) = P[\chi^2_1 > t]/2$.

<u>Proof</u>. By the continuity of $\pi_{12}$, $\inf_{\mu\notin H_1} \pi_{12}(\mu) = \inf_{\mu\notin C_1} \pi_{12}(\mu)$. Fix $i > 1$, then there is some $j$ with $x_j^* = x_{j+1}^*$ for all $x \in C_i$ and $C_{j+1} = \{x : x_1^* < \cdots < x_j^* = x_{j+1}^* < \cdots < x_k^*\}$. Hence, $C_i' \subset C_{j+1}'$ $= \{x \in R^k : x_j \leq x_{j+1}\}$, $\|E_W(x \mid C_i')-x\|_W \geq \|E_W(x \mid C_{j+1}')-x\|_W$ for all $x \in R^k$ and $P_{0,W}[\|E_W(x \mid C_i')-x\|^2_W > t] \geq P_{0,W}[\|E_W(x \mid C_{j+1}')-x\|_W > t]$. So, $\inf_{\mu\notin H_1} \pi_{12}(\mu) = \inf_{1\leq j\leq k} P_{0,W}[\|E_W(x \mid C_{j+1}')-x\|^2_W > t]$. But, $E_W(x \mid C_{j+1}')_i$ $= x_i$ for $i \neq j,j+1$ and $(E_W(x \mid C_{j+1}')_j, E_W(x \mid C_{j+1}')_{j+1})$ is the projection of $(x_j,x_{j+1})$ onto $\{y \in R^2 : y_1 \leq y_2\}$ with norm defined by $\|y\|^2 = w_j y_1^2 + w_{j+1} y_2^2$. Using Corollary 4.2 of Robertson and Wegman (1978) and the fact that for a total order and any weights $P(1,2) = P(2,2) = 1/2$, we see that $P_{0,W}[\|E_W(x \mid C_{j+1}')-x\|_W > t] = P[\chi^2_1 > t]/2$. The proof is completed.

It is of interest to examine the "amount" of bias in $T_{12}$ as $k$ increases. So with $t$ the 5% critical value for $T_{12}$ with $w = e_k$ and $k = 3,4,5,6$, we computed the infimum in Corollary 3.16. As was noted before, for $k = 3$ the infimum is .01620, for $k = 4$ it is .00648, for $k = 5$ it is .00281 and for $k = 6$ it is .00128. (The critical values are taken from Robertson and Wegman (1978)). The infimum is approximated by $\pi_{12}(\mu)$ with $\mu$ at a large distance from $H_0$, but close to $H_1$. For practical purposes it is also of interest to compute $\pi_{12}$ at $\mu$ near $H_1$, but at a "reasonable" distance from $H_0$. We first consider $k = 3$ and $w = e_3$, for in this case the powers can be obtained numerically. Because of the proof of Corollary 3.16, we will compute $\pi_{12}(\nu + \alpha E(\nu \mid H_1))$ for various $\alpha$ and $\nu$ chosen so that $E(\nu \mid H_1)$ has one level set with two elements and the other has one element. Table 1 gives the values of $\pi_{12}(\nu_i + \alpha E(\nu_i \mid H_1))$ for $\nu_i = \mu_i/\Delta(\mu_i)$ with $\mu_1 = (2,1,2)$ and $\mu_2 = (1.5,1,2)$ and $t = 4.578$, the 5% critical value.

We observe that the bias of $T_{12}$, even for $k = 3$, is large enough to be of practical significance. For $k = 5$, $w = e_5$, $t = 7.665$ (the 5% critical value of $T_{12}$) and $\nu = \mu/\Delta(\mu)$ with $\mu = (3,1,3,4,5)$, $\pi_{12}(\nu + \alpha E(\nu \mid H_1))$ are estimated by Monte Carlo techniques with 10,000 replications. These values are given in Table 2. We notice that the bias is even more pronounced for $k = 5$.

TABLE 1. Values of $\pi_{12}(\nu_i+\alpha E(\nu_i | H_1))$ with $k = 3$, $w = e_3$, $t = 4.578$ and $\nu_i = \mu_i/\Delta(\mu_i)$.

| | $\mu_1 = (2,1,2)$ | | | $\mu_2 = (1.5,1,2)$ | | |
|---|---|---|---|---|---|---|
| $\alpha$ | $\nu_1+\alpha E(\nu_1 \| H_1)$ | $\Delta$ | $\pi_{12}$ | $\nu_2+\alpha E(\nu_2 \| H_1)$ | $\Delta$ | $\pi_{12}$ |
| -1 | (.6124,-6124,0) | .866 | .1448 | (.3536,-.3536,0) | .500 | .0880 |
| 0 | (2.449,1.225,2.449) | 1.000 | .1338 | (2.121,1.414,2.828) | 1.000 | .0481 |
| 1 | (4.287,3.062,4.899) | 1.323 | .0770 | (3.889,3.182,5.657) | 1.803 | .0272 |
| 2 | (6.124,4.899,7.348) | 1.732 | .0466 | (5.657,4.950,8.485) | 2.646 | .0227 |
| 5 | (11.64,10.41,14.70) | 3.123 | .0270 | (10.96,10.25,16.97) | 5.220 | .0192 |
| 10 | (20.82,19.60,26.94) | 5.568 | .0213 | (19.80,19.09,31.11) | 9.539 | .0178 |

TABLE 2. Values of $\pi_{12}(\nu+\alpha E(\nu | H_1))$ with $k = 5$, $w = e_5$, $t = 7.665$ and $\nu = \mu/\Delta(\mu)$.

| | $\mu = (3,1,3,4,5)$ | | |
|---|---|---|---|
| $\alpha$ | | $\Delta$ | $\pi_{12}$ |
| -1 | (.3371,-3371,0,0,0) | .48 | .0699 |
| 0 | (1.011,.3371,1.011,1.348,1.686) | 1.00 | .0334 |
| 1 | (1.686,1.011,2.023,2.700,3.371) | 1.82 | .0204 |
| 2 | (2.360,1.686,3.034,4.045,5.057) | 2.68 | .0151 |
| 5 | (4.384,3.708,6.068,8.090,10.11) | 5.30 | .0108 |
| 10 | (7.753,7.079,11.12,14.83,18.54) | 9.68 | .0101 |

4. <u>CONTRAST TESTS</u>. Suppose one is to test $H_0$ versus $H_1-H_0$ with a contrast test which rejects for large values of $T_c = \sum_{i=1}^k w_i c_i \bar{X}_i$ with $w_i = n_i/\sigma_i^2$ and $C \neq 0$. Assuming the weights, $w_i$, are equal, Abelson and Tukey (1963) found that the optimal contrast coefficients are $c_i^{(0)} \propto ((i-1)(k-i+1))^{1/2} - (i(k-i))^{1/2}$, $1 \le i \le k$. Schaafsma and Smid (1966) generalized their work to the case of unequal weights and obtained

$$(4.1) \qquad w_i c_i^{(0)} \propto (s_{i-1}(s_k - s_{i-1}))^{1/2} - (s_i(s_k - s_i))^{1/2}$$

$$\text{with } s_i = \sum_{j=1}^i w_j \text{ and } s_0 = 0.$$

We note that $\sum_{i=1}^k w_i c_i^{(0)} = 0$ and so the distribution of $T_{c^{(0)}}$ is the same for all $\mu \in H_0$.

One could also consider testing $H_1$ versus $H_2$ by rejecting for large values of such a statistic. Of course the contrast coefficients for testing $H_1$ versus $H_2$ would be different than those chosen for testing $H_0$ versus $H_1-H_0$. The power function for the test (whether testing $H_0$ versus $H_1-H_0$ or $H_1$ versus $H_2$) is given by

$$(4.2) \qquad \pi_c(\mu) = 1 - \Phi((t-(c,\mu)_W)/\|c\|_W).$$

Since the distribution of $T_c$ may not be the same for all $\mu \in H_1$, the level of significance is $\sup_{\mu \in H_1} \pi_c(\mu)$. If there is a $\mu \in H_1$ with $(c,\mu)_W > 0$, then using the fact that $H_1$ is a cone, we see that this supremum is 1. Thus, we restrict attention to $c$ with $(\mu,c)_W \le 0$ for all $\mu \in H_1$, or equivalently $c \in H_1^{*W}$. For such $c$, the level of significance is $\sup_{\mu \in H_1} \{1 - \Phi((t-(c,\mu)_W)/\|c\|_W)\} = 1 - \Phi(t/\|c\|_W)$. Thus, if $z_p$ satisfies $\Phi(z_p) = 1-p$, then $t = z_p\|c\|_W$ gives a test of size $p$.

We now consider different optimality criteria and the corresponding $c$. Fix $\mu \notin H_1$ and consider the contrast test which maximizes the power at $\mu$, that is, $c$ maximizes $(c,\mu)_W/\|c\|_W$ over all $c \in H_1^{*W} - \{0\}$. Since $\mu \notin H_1$, there eixsts a $j$ with $\mu_j > \mu_{j+1}$. Consider $c$ with $c_i = 0$ for $i \neq j,j+1$ and $c_j = -c_{j+1} = 1$, then $\rho(c,\mu) > 0$. If we agree that $(0,\mu)_W/\|0\|_W = 0$, then the maximization problem is unchanged if $H_1^{*W} - \{0\}$ is replaced by $H_1^{*W}$. Since $\mu$ is fixed and $\tilde{c} = 0$ for $c \in H_1^{*W}$, the above is equivalent to

(4.3) $\quad$ maximize $\rho(c,\mu) = \sum_{i=1}^{k} w_i(c_i-\tilde{c})(\mu_i-\tilde{\mu})/(\|c-\tilde{c}\|_W\|\mu-\tilde{\mu}\|_W)$ with $c \in H_1^{*W}$

(set $\rho(0,\mu) = 0$). Clearly, $H_1^{*W} \oplus H_0$, which is characterized in (3.3), is a closed convex cone containing the constant functions. Furthermore, $c$ solves maximize $\rho(c,\mu)$ over $c \in H_1^{*W} \oplus H_0$ if and only if $c-\tilde{c}$ solves (4.3). Applying (ii) of Corollary E, p. 320 of Barlow et al. (1972), $E_W(\mu \mid H_1^{*W} \oplus H_0)$ maximizes $\rho(c,\mu)$ for $c \in H_1^{*W} \oplus H_0$. Using (2.1) it is easily shown that $E_W(\mu \mid H_1^{*W} \oplus H_0) = E_W(\mu \mid H_1^{*W}) + \tilde{\mu}$. Since $\sum_{i=1}^{k} w_i(E_W(\mu \mid H_1^{*W})_i + \tilde{\mu})/\sum_{i=1}^{k} w_i = \tilde{\mu}$, $E_W(\mu \mid H_1^{*W})$ solves (4.3). The power function of the resulting test is $1 - \Phi(z_p-(E_W(\mu \mid H_1^{*W})_W,\mu)/\|E_W(\mu \mid H_1^{*W})\|_W)$, which by (2.2) can be written as $1 - \Phi(z_p-\|E_W(\mu \mid H_1^{*W})\|_W)$. We have proved:

<u>Theorem 4.1</u>. Let $\mu \notin H_1$. The contrast test with maximum power at $\mu$ is determined by $c = E_W(\mu \mid H_1^{*W})$. The power function is $\pi_c(\mu) = 1 - \Phi(z_p-\|E_W(\mu \mid H_1^{*W})\|_W)$.

Since the optimum $c$ depends on the unknown $\mu$, one could estimate $c$ using $E_W(\bar{X} \mid H_1^{*W}) = \bar{X}-E_W(\bar{X} \mid H_1)$. However, $\sum_{i=1}^{k} w_i(\bar{X}_i - E_W(\bar{X} \mid H_1)_i)\bar{X}_i = \|\bar{X} - E_W(\bar{X} \mid H_1)\|_W^2 = T_{12}$ (cf. (2.1)). Thus, $T_{12}$ is an adaptive contrast test.

Next, we consider the criterion used by Abelson and Tukey (1963), that is, we fix $\delta > 0$ and seek contrast coefficients which maximize the minimum power over points at distance $\delta$ from the null hypothesis, $H_1$. So, we wish to solve

$$\sup_{c \in H_1^{*W} - \{0\}} \inf_{\mu : \|\mu - E_W(\mu | H_1)\|_W = \delta} \{1 - \Phi(z_p - (c,\mu)_W / \|c\|_W)\}.$$

However, we will show that for $c \in H_1^{*W} - \{0\}$ and $\delta > 0$, $\inf_{\mu : \|\mu - E_W(\mu | H_1)\|_W = \delta} \pi_c(\mu) = 0$ so that this criterion is not useful.

<u>Lemma 4.2</u>. If $c \in H_1^{*W} - \{0\}$, $\delta > 0$ and $k > 2$, then there exists a $\mu \notin H_1$ with $\|\mu - E_W(\mu | H_1)\|_W = \delta$ and $(c, E_W(\mu | H_1))_W < 0$.

<u>Proof</u>. Let $v_1 = (w_1^{-1}, -w_2^{-1}, 0, \cdots, 0)$, $v_2 = (0, w_2^{-1}, -w_3^{-1}, 0, \cdots, 0)$, $\cdots, v_{k-1} = (0, \cdots, 0, w_{k-1}^{-1}, -w_k^{-1})$. It is easy to show that $H_1^{*W} = \{a_1 v_1 + \cdots + a_{k-1} v_{k-1} : a_i \geq 0\}$. Furthermore, $(v_i, E_W(\mu | H_1))_W \leq 0$ for each $i$ and $\mu$. Let $c = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$. If $a_j > 0$, $1 \leq j < k-1$, then let $\mu$ be $(1, 2, \cdots, k)$ with the $j+1^{\underline{st}}$ and $j+2^{\underline{nd}}$ coordinates interchanged. So, $E_W(\mu | H_1)_i = i$ for $i \neq j+1, j+2$ and $E_W(\mu | H_1)_i = ((j+2)w_{j+1} + (j+1)w_{j+2})/(w_{j+1} + w_{j+2})$ for $i = j+1, j+2$. Thus, $a_j(v_j, E_W(\mu | H_1))_W = a_j(j - E_W(\mu | H_1)_{j+1}) < 0$ and $(c, E_W(\mu | H_1))_W < 0$. If $a_1 = \cdots = a_{k-2} = 0$, and $a_{k-1} > 0$ (recall, $c \neq 0$), then let $\mu = (2, 1, \cdots, k)$. It is easy to show that $a_{k-1}(v_{k-1}, E_W(\mu | H_1))_W < 0$. Thus, in either case, one can find $\mu \notin H_1$ with $(c, E_W(\mu | H_1))_W < 0$. Multiplying by the appropriate positive constant, we obtain $\mu \notin H_1$ with $(c, E_W(\mu | H_1))_W < 0$ and $\|\mu - E_W(\mu | H_1)\|_W = \delta$. The proof is completed.

For $a \geq -1$, set $\mu_a = \mu + a E_W(\mu \mid H_1)$ and note that by the remark preceding Theorem 3.1, $\|\mu_a - E_W(\mu_a \mid H_1)\|_W = \|\mu - E_W(\mu \mid H_1)\|_W = \delta > 0$. Thus, $\mu_a \notin H_1$, the distance from $\mu_a$ to $H_1$ is $\delta$ and

$$\lim_{a\to\infty}(c,\mu_a)_W = (c,\mu)_W + \lim_{a\to\infty} a(c, E_W(\mu \mid H_1))_W = -\infty$$

Therefore, for each $c \in H_1^{*W} - \{0\}$ and $\delta > 0$,

$$\inf_{\mu:\|\mu - E_W(\mu \mid H_1)\|_W = \delta} \pi_c(\mu) = 0.$$

We must consider other criteria.

Following Schaafsma and Smid (1966), we consider the contrast that minimizes the maximum "shortcoming" among all contrast tests. Recall that for a given $\mu \notin H_1$, the contrast test with maximum power at $\mu$ is obtained by taking $c = \mu - E_W(\mu \mid H_1)$ and has power $1 - \Phi(z_p - \|\mu - E_W(\mu \mid H_1)\|_W)$. So, for any contrast test its shortcoming at $\mu$ is

$$(4.4) \qquad \Phi(z_p - (c,\mu)_W/\|c\|_W) - \Phi(z_p - \|\mu - E_W(\mu \mid H_1)\|_W).$$

If there is no constraint on $\mu$ other than $\mu \notin H_1$, we see from the preceding analysis that the supremum is at least as large as $1 - \Phi(z_p - \delta)$ for each $\delta > 0$, and so the maximum shortcoming over all $\mu \notin H_1$ is 1. Even if $\mu$ is constrained so that $\|\mu - E_W(\mu \mid H_1)\|_W = \delta > 0$, the maximum shortcoming is $1 - \Phi(z_p - \delta)$ which does not depend on $c$. Neither of these criteria are useful.

The vector of means $\mu + aE_W(\mu \mid H_1)$ remains at a fixed distance from $H_1$, but it is moving away from $H_0$ as $a$ increases. So, we consider the contrast test which maximizes the minimum power over all $\mu \notin H_1$ with $\Delta(\mu) = \|\mu - \tilde{\mu}\|_W = \delta > 0$. Let $a_i = (w_i^{-1} + w_{i+1}^{-1})^{1/2}$ for $i = 1,2,\cdots,k-1$, let $d_1' = 0$, $d_i' = \sum_{j=1}^{i-1} a_j$ for $i = 2,\cdots,k$, let $d_1 = d' - \tilde{d}'$ and let $c^{(1)} = -d_1$.

<u>Theorem 4.3</u>. Let $\delta > 0$. The contrast test which has coefficients $c^{(1)}$ and rejects for large values of $T_{c^{(1)}}$ maximizes the minimum power over all $\mu \notin H_1$ with $\Delta(\mu) = \delta$. Furthermore, such contrast coefficients are unique up to a positive multiplier.

Before the proof of the theorem is given, we establish

<u>Lemma 4.4</u>. If $\mu,\nu \in H_1$, then $(\mu-\tilde{\mu},\nu-\tilde{\nu})_W \geq 0$.

<u>Proof</u>. $\tilde{\mu}-\mu \in H_1^{*W}$ and $\nu-\tilde{\nu} \in H_1$ and the conclusion is immediate.

<u>Proof of Theorem 4.3</u>. We wish to find $c$ which yields $\sup_{c\in H_1^{*W}-\{0\}}$ $\inf_{\mu\notin H_1,\Delta(\mu)=\delta}\{1-\Phi(z_p-(c,\mu)_W/\|c\|_W)\}$, or since $\tilde{c} = 0$, equivalently,

(4.5) $\quad \sup_{c\in H_1^{*W}-\{0\}} \inf_{\mu\notin H_1} \rho(c,\mu).$

If $-c \notin H_1$, then consider $\mu = -c$. Since $\rho(c,-c) = -1$, we may omit such $c$ from the supremum. Because $H_1^{*W} \cap (-H_1) = (-H_1) \cap \{\mu : \tilde{\mu} = 0\}$, (4.5) is equivalent to

(4.6) $\quad \sup_{-c\in H_1-\{0\},\tilde{c}=0} \inf_{\mu\notin H_1} \rho(c,\mu) = -\inf_{d\in H_1-\{0\},\tilde{d}=0} \sup_{\mu\notin H_1} \rho(d,\mu).$

We will solve for $d$ and remember that $c = -d$. Because of the continuity of $\rho(d,\cdot)$, the supremum in the r.h.s. could also be taken over $\mu \notin H_1^0 \cup H_0$. However, if $\mu \in \partial H_1$, then applying Lemma 4.4, $\rho(d,\mu) \geq 0$ and so that supremum could be restricted to $\mu \notin H_1^0 \cup (H_0 \oplus H_1^{*W})$ (for $\rho(d,\mu) \leq 0$

for $\mu \in H_1^{*W}$). So, we seek $d$ which solves

$$(4.7) \qquad \inf_{d\in H_1-\{0\},\tilde{d}=0} \sup_{\mu\notin H_1^0 \cup (H_0 \oplus H_1^{*W})} \rho(d,\mu).$$

Furthermore, if $\mu \notin H_1^{*W} \oplus H_0$, then $E_W(\mu \mid H_1) \notin H_0$ and so $\Delta(E_W(\mu \mid H_1)) > 0$. Applying (2.3) and the fact that $\sum_{i=1}^k w_i E_W(\mu \mid H_1)_i = \sum_{i=1}^k w_i\mu_i$, we see that $0 < \Delta(E_W(\mu \mid H_1)) = \|E_W(\mu \mid H_1)-\tilde{\mu}\|_W = \|E_W(\mu-\tilde{\mu} \mid H_1)\|_W \le \|\mu-\tilde{\mu}\|_W = \Delta(\mu)$. For fixed $d \in H_1-\{0\}$ with $\tilde{d} = 0$, $\rho(d,\mu) = (\|d\|_W \cdot \|\mu-\tilde{\mu}\|_W)^{-1}(d,\mu)_W$ $\le (\|d\|_W \|\mu-\tilde{\mu}\|_W)^{-1}(d,E_W(\mu \mid H_1))_W$, which is nonnegative by Lemma 4.4. So $\rho(d,\mu) \le \rho(d,E_W(\mu \mid H_1))$ for $\mu \notin H_1^{*} \oplus H_0$. Therefore, $d$ solves

$$\inf_{d\in H_1-\{0\},\tilde{d}=0} \sup_{\mu\in H_1^0 \cup (H_1^{*W}\oplus H_0)} \rho(d,E_W(\mu \mid H_1))$$
$$= \inf_{d\in H_1-\{0\},\tilde{d}=0} \sup_{\mu\in\partial H_1-H_0} \rho(d,\mu).$$

The boundary of $H_1$ is the union of $A_1 = \{x \in R^k : x_1 = x_2 \le x_3 \le \cdots \le x_k\}$, $A_2 = \{x \in R^k : x_1 \le x_2 = x_3 \le \cdots \le x_k\},\cdots,A_{k-1} = \{x \in R^k : x_1 \le \cdots \le x_{k-1} = x_k\}$. Because of the convention $\rho(d,0) = 0$, we seek $d$ that solves

$$(4.8) \qquad \inf_{d\in H_1-\{0\},\tilde{d}=0} \max_{1\le i\le k-1} \max_{\mu\in A_i} \rho(d,\mu).$$

Each $A_i$ is a closed, convex cone in $R^k$ containing the constant functions and $\rho(d,\mu) \ge 0$ for any $\mu \in A_i$. So, by Corollary E, p. 320 of Barlow et al. (1972), $\max_{\mu\in A_i} \rho(d,\mu) = \rho(d,E_W(d \mid A_i))$. It is easy to show that $d^* = (d_1^*,\cdots,d_k^*)$, with $d_j^* = d_j$ for $j \ne i,i+1$ and $d_j^* = (w_i d_i + w_{i+1} d_{i+1})/(w_i + w_{i+1})$ for $j = i,i+1$, is the point in $A_i$ closest to $d \in H_1$, i.e., $d^* = E_W(d \mid A_i)$. Also

$\rho(d,E_W(d \mid A_i)) = \|E_W(d \mid A_i)\|_W/\|d\|_W$. So $d$ solves (4.8) if and only if $d/\|d\|_W$ solves

$$\min_{d\in H_1, \|d\|_W=1, \tilde{d}=0} \max_{1\le i\le k-1} \|E_W(d \mid A_i)\|_W.$$

However, $\|d\|_W - \|E_W(d \mid A_i)\|_W = w_i w_{i+1}(d_{i+1}-d_i)^2/(w_i+w_{i+1}) = (d_{i+1}-d_i)^2/a_i^2$. So, we wish to solve

$$(4.9) \qquad \max_{d\in H_1, \|d\|_W=1, \tilde{d}=0} \min_{1\le i\le k-1} (d_{i+1}-d_i)^2/a_i^2.$$

Let $d_1$ be defined as in the paragraph before the statement of the theorem and let $d_a = ad_1$. Note that $d_a \in H_1$, $\tilde{d}_a = 0$ and $\|d_a\|_W = a \cdot \|d_1\|_W > 0$ for all $a > 0$. So, choose $a$ so that $\|d_a\|_W = 1$.

We now show that the $d_a$ chosen above is the unique solution to (4.9), which implies that $-d_a$ is the unique, up to a positive multiplier, set of contrast coefficients which is being sought. Note that if $d_a = (d_{a1}, d_{a2}, \cdots, d_{ak})$, then $(d_{ai+1} - d_{ai})^2/a_i^2 = a^2$ for $i = 1,2,\cdots,k-1$. Suppose $z \in H_1$ with $\tilde{z} = 0$, $\|z\|_W = 1$ and $\min_{1\le i\le k-1}(z_{i+1}-z_i)^2/a_i^2 \ge a^2$. Then, $(z_{i+1}-z_i) \ge (d_{ai+1}-d_{ai})$ or $z_{i+1}-d_{ai+1} \ge z_i-d_{a_i}$ for $i = 1,2,\cdots,k-1$. Hence, $z-d_a \in H_1$ and applying Lemma 4.4,

$$1 = \|3\|_W^2 = \|d_a\|_W^2 + \|z-d_a\|_W^2 + 2(d_a, z-d_a)_W \ge 1+\|z-d_a\|_W^2.$$

So, $\|z-d_a\|_W^2 = 0$ or $z = d_a$. The proof is completed.

We conclude this section with some remarks concerning the power functions of such contrast tests, that is, tests which reject for large values of $T_c$.

Theorem 4.5. Let $\mu,\nu \in R^k$. If $(c,\nu)_W = 0$, then $\pi_c(\mu+\alpha\nu) = \pi_c(\mu)$ for all $\alpha \in (-\infty,\infty)$. If $(c,\nu)_W > 0$ $((c,\nu)_W < 0)$, then $\pi_c(\mu+\alpha\nu)$ is increasing (decreasing) in $\alpha$ with $\lim_{\alpha\to\infty} \pi_c(\mu+\alpha\nu) = 1$ (0) and $\lim_{\alpha\to-\infty} \pi_c(\mu+\alpha\nu) = 0$ (1).

Proof. The result follows immediately from (4.2) since $(c,\mu+\alpha\nu)_W = (c,\mu)_W + \alpha(c,\nu)_W$.

In the next result the regions of consistency are determined for such contrast tests.

Theorem 4.6. Let $\mu,\gamma \in R^k$ with $\gamma_i > 0$ for $i = 1,2,\cdots,k$. Let $w_n = \eta\gamma$ and fix the level of the contrast test at $p \in (0,1)$ for all $n$. If $(c,\mu)_\gamma > 0$ $((c,\mu)_\gamma < 0)$, then $\pi_c(\mu) \to 1$ (0) as $n \to \infty$. If $(c,\mu)_\gamma = 0$, then $\pi_c(\mu) = p$ for all $n$.

Proof. Since $(c,\mu)_{W_n}/\|c\|_{W_n} = n^{1/2}(c,\mu)_\gamma/\|c\|_\gamma$, the desired conclusion follows from (4.2).

It is of interest to compare the regions of consistency for $T_{01}$ and $T_{c^{(0)}}$ in testing $H_0$ versus $H_1 - H_0$. We first show that $c^{(0)} \in H_1^0$. Let $x_i = s_i/s_k$ for $i = 0,1,\cdots,k$ and note that (cf. (4.1)) $c_i^{(0)} \propto (g(x_{i-1})-g(x_i))/(x_i-x_{i-1})$ where $g(x) = (x(1-x))^{1/2}$. The desired conclusion follows from the strict concavity of $g$ on $[0,1]$. Applying Theorem 4.6 the contrast test is consistent for $\mu \in A\,(c^{(0)}) = \{\mu : (c^{(0)},\mu)_W > 0\}$, and from Corollary 3.2, this is true for $T_{01}$ for $\mu \notin H_1^{*W} \oplus H_0$. If $\mu = \mu' + \mu''$ with $\mu' \in H_1^{*W}$ and $\mu'' \in H_0$, then $(c^{(0)},\mu)_W = (c^{(0)},\mu')_W \le 0$. So, $A^+(c^{(0)}) \subset (H_1^{*W} \oplus H_0)^c$. Drawing

$H_1$, $H_1^{*W}$, $c^{(0)}$ and $A^+(c^{(0)})$ for $k = 3$ in the plane $\tilde{\mu} = 0$ gives one an idea of the size of $(H_1^{*W} \oplus H_0)^c - A^+(c^{(0)})$.

In testing $H_1$ versus $H_2$, $T_{c^{(1)}}$ is consistent for $\mu \in A^+(c^{(1)})$ and $T_{12}$ is consistent for $\mu \notin H_1$. Since $-c^{(1)} \in H_1$ and $\tilde{c}^{(1)} = 0$, $c^{(1)} \in H_1^{*W}$. If $\mu \in H_1$, then $(c^{(1)}, \mu)_W \leq 0$ and $\mu \in (A^+(c^{(1)}))^c$. Hence, $A^+(c^{(1)}) \subset H_1^c$. Again one can obtain an idea of the size of $H_1^c - A^+(c^{(1)})$ by drawing the figure for $k = 3$.

5. COMMENTS. We begin this section with a few remarks about the situation in which the variances are unknown. Suppose $X_{ij}$ are independent $\eta(\mu_i,\sigma^2)$ variables for $j = 1,2,\cdots,n_i$ and $i = 1,2,\cdots,k$ with $\mu = (\mu_1,\mu_2,\cdots,\mu_k)$ and $\sigma^2$ unknown. For the contrast tests, let $w_i = n_i$ and assume that $c$ does not depend on $\sigma^2$. Following the optimal procedure for $k = 2$, define $\hat{\sigma}^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij}-\overline{X}_i)^2/(N-k)$ with $N = \sum_{i=1}^{k} n_i$. In testing $H_0$ versus $H_1-H_0$, we assume $\sum_{i=1}^{k} w_i c_i = 0$, and so if one rejects for large $T_c' = \sum_{i=1}^{k} w_i c_i \overline{X}_i/\hat{\sigma}$, then the $100p\%$ critical value is $t_{N-k,p}\|c\|_W$ where $F(t_{N-k,p}) = 1-p$ with $F$ the distribution function for a Students $t$ variable with $N-k$ degrees of freedom. In testing $H_1$ versus $H_2$, we assume $c \in H_1^{*W}$, and so $H_0$ is least favorable within $H_1$. Hence, the $100p\%$ critical value is also $t_{N-k,p}\|c\|_W$. Let $f(y)$ be the density of $Y_n = \hat{\sigma}/\sigma$. Conditioning on $Y_N$, which is independent of $\overline{X} = (\overline{X}_1,\overline{X}_2,\cdots,\overline{X}_k)$, we see that the modified contrast test has power function

$$(5.1) \qquad \pi_c'(\mu) = 1 - \int_0^{\infty} \Phi(yt_{N-k,p} - (c,\mu)_W/(\sigma\|c\|_W))f(y)dy.$$

Hence, Theorem 4.5, which gives the radial monotonicity and radial limits of $\pi_c$, is also valid for $\pi_c'$. Furthermore, if $n_i = n\gamma_i$ with $\gamma_i > 0$ for $i = 1,2,\cdots,k$, then as $n \to \infty$, $\sum_{i=1}^{k} n_i c_i \overline{X}_i/\|c\|_W = \sqrt{n} \sum_{i=1}^{k} \gamma_i c_i \overline{X}_i/\|c\|_\gamma \to \pm\infty$ depending on whether $(c,\mu)_\gamma > 0$ or $(c,\mu)_\gamma < 0$. Also $(\hat{\sigma}/\sigma)t_{N-k,p} \to z_p$. So as $n \to \infty$, $\pi_c'(\mu) \to 1$ $(0)$ as $n \to \infty$ provided $(c,\mu)_\gamma > 0$ $((c,\mu)_\gamma < 0)$, and $\pi_c'(\mu) \to p$ if $(c,\mu)_\gamma = 0$. The radial behavior and the regions of consistency for these modified contrast tests are like those for the contrast tests.

The LRT for $H_0$ versus $H_1-H_0$ rejects for large values of

$$S_{01} = \Sigma_{i=1}^{k} n_i(\bar{\mu}_i-\hat{\mu})^2/\Sigma_{i=1}^{k} \Sigma_{j=1}^{n_i}(\bar{X}_{ij}-\hat{\mu})^2 = \sigma^2 T_{01}/S_T$$

where $S_T$ is the total sum of squares (cf. Barlow et al. (1972), p. 121)). Equivalently, one could reject for large values of $L_{01} = (N-k)S_{01}/(1-S_{01})$. But $S_T = \Sigma_{i=1}^{k} \Sigma_{j=1}^{n_i}(X_{ij}-\bar{\mu}_i)^2 + \Sigma_{i=1}^{k} n_i(\bar{\mu}_i-\hat{\mu})^2 + 2\Sigma_{i=1}^{k} n_i(\bar{\mu}_i-\hat{\mu})(\bar{X}_i-\mu_i)$. Applying (2.1) and the fact that $\Sigma_{i=1}^{k} w_i E_W(x \mid c)_i = \Sigma_{i=1}^{k} w_i x_i$, we see that the last sum is zero. Hence, $L_{01} = (N-k)\Sigma_{i=1}^{k} n_i(\bar{\mu}_i-\hat{\mu})^2/\Sigma_{i=1}^{k} \Sigma_{j=1}^{n_i}(X_{ij}-\bar{\mu}_i)^2$. We now wish to determine the region of consistency for $L_{01}$. Suppose that $n_i = n\gamma_i$ with $\gamma_i > 0$. Noting that $\Sigma_{i=1}^{k} \Sigma_{j=1}^{n_i}(X_{ij}-\bar{\mu}_i)^2/(N-k)$ $= \Sigma_{i=1}^{k} \Sigma_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2/(N-k) + \Sigma_{i=1}^{k} n_i(\bar{X}_i-\bar{\mu}_i)^2/(N-k)$, we see that this expression converges almost surely to $\sigma^2 + \|\mu - E_\gamma(\mu \mid H_1)\|_\gamma^2/\Sigma_{i=1}^{k} \gamma_i$. Under $H_0$ (we may assume without loss of generality that $\mu = 0$), $\Sigma_{i=1}^{k} n_i(\bar{\mu}_i-\hat{\mu})^2$ $\overset{\mathcal{D}}{=} \sigma^2\Sigma_{i=1}^{k} \gamma_i(E_\gamma(Y \mid H_1)_i - \Sigma_{j=1}^{k} \gamma_j Y_j/\Sigma_{j=1}^{k} \gamma_j)^2$ where $Y_1, Y_2, \cdots, Y_k$ are independent variables with $Y_i \sim \eta(0,\gamma_i^{-1})$. Hence, the 100p% critical value for $L_{01}$ converges to the 100p% critical value for $T_{01}$ with weights $\gamma_i$.

Now suppose $\mu \in H_1^{*W} \oplus H_0$. Examining the proof given in Barlow et al. (1972) for their Theorem 3.4, we see that $\Sigma_{i=1}^{k} n_i(\bar{\mu}_i-\hat{\mu}))^2 \xrightarrow{a.s.} \infty$ and, as we have seen, $\Sigma_{i=1}^{k} \Sigma_{j=1}^{n_i}(X_{ij}-\bar{\mu}_i)^2/(N-k) \xrightarrow{a.s.} \sigma^2 + \|\mu-E_\gamma(\mu \mid H_1)\|_\gamma^2/\Sigma_{i=1}^{k} \gamma_i$. Hence, $L_{01}$ is consistent for such $\mu$. Furthermore, the argument given in the second part of their proof shows that $\Sigma_{i=1}^{k} n_i(\bar{\mu}_i-\hat{\mu})^2 \xrightarrow{a.s.} 0$ provided $\mu \in (H_1^{*W} \oplus H_0)^0$. This is the same limiting behavior as was observed for $T_{01}$.

It is interesting to note that

$$L_{01} = \frac{T_{01}}{Q + T_{12}}$$

where $Q$ is independent of $T_{01}$ and $T_{12}$. Recall from the introduction that $T_{01}$ is isotonic and $T_{12}$ is antitonic with respect to $\lesssim$. It follows that for fixed $q$, $P_\mu\left[\frac{T_{01}}{q+T_{12}} > t\right]$ is isotonic with respect to $\lesssim$ and by conditioning that $P_\mu[L_{01} > t]$ is isotonic with respect to $\lesssim$. Thus, if $\mu \in H_1$ then $P_{\delta\mu}[L_{01} > t]$ is nondecreasing for $\delta \in (-\infty,\infty)$ and $P_{\mu+\delta E_W(\mu|H_1)}[L_{01} > t]$ is nondecreasing for $\delta \in (-\infty,\infty)$ for any $\mu \in R_k$. What about directions like $\mu+\delta E_W(\mu \mid H_1^*)$ or for $\mu \in (-H_1^*) \cup H_0$? It is easy to see that for fixed $q > 0$, $\left\{x \in R^k : \frac{\|E_W(x|C_{01})\|_W^2}{q + \|E_W(x|C_{12})\|_W^2} \le t\right\}$ is not convex (take $q = t = 1$ and $k = 3$) so that the techniques of Section 3 will not apply.

The LRT for $H_1$ versus $H_2$ rejects for large values of

$$S_{12} = \Sigma_{i=1}^k n_i(\bar{X}_i-\bar{\mu}_i)^2/[(N-k)\hat{\sigma}^2 + \Sigma_{i=1}^k n_i(\bar{X}_i-\bar{\mu}_i)^2]$$

(cf. Robertson and Wegman (1978)), or equivalently for large values of

$$L_{12} = (N-k)S_{12}/(1-S_{12}) = \Sigma_{i=1}^k n_i(\bar{X}_i-\bar{\mu}_i)^2/\hat{\sigma}^2 = \sigma^2 T_{12}/\hat{\sigma}^2.$$

If we again denote the density of $Y_N = \hat{\sigma}/\sigma$, by $f(y)$, then, for a fixed critical value, $t > 0$, the power function of $L_{12}$ is given by

$$\pi'_{12}(\mu) = \int_0^\infty P_{\mu,W}[\sqrt{T_{12}} > \sqrt{t}\, y]f(y)dy.$$

Hence, Theorem 3.7, Corollary 3.8 and Corollary 2.11 are also valid for $\pi'_{12}$. It is not difficult to show that $S_{12}$ is consistent for all $\mu \notin H_1$, as was the case for $T_{12}$.

Because of the similarities we have observed between the case of variances known and the case of variances unknown, one might conjecture that the monotonicity properties of $L_{01}$ are like those of $T_{01}$. However, we have seen that some of the results do not follow from simple conditioning arguments as in the case of $T_{12}$ and $L_{12}$. It would be of interest to know what techniques could be applied in the study of the monotonicity of the power function of $L_{01}$.

In deriving the optimal contrast test for $H_1$ versus $H_2$, the vector $d_1$ was obtained. This vector, which in the case of equal weights has uniform increments (i.e., $d_{1,i+1}-d_{1,i}$ is constant), is in the center of the cone $H_1$. In fact, $d_1$ makes equal angles with the faces of $H_1$. On the other hand, the optimal conrast test of $H_0$ versus $H_1-H_0$ is based on $c_{(0)}$, which is another center of $H_1$. The vector $c_{(0)}$ makes equal angles with the edges of $H_1$. Bartholomew (1961) conjectured that, for a fixed value of $\Delta$, the power of $T_{01}$ is largest at $d_1$. It is of interest to compare the power of $T_{01}$ at both of the "centers" mentioned above. Fixing their lengths to be 1 and $k = 5$, $d_1 = (-.6325,-.3162,0,.3162,.6324)$ and $c_{(0)} = (-.6899,-.1551,0,.1551,.6899)$. For $w = e_5$, these powers were estimated by a Monte Carlo experiment with 9,999 replications. The estimates are $\pi_{01}(d_1) = .2374$ and $\pi_{01}(c_{(0)}) = .2339$, which tends to confirm Bartholomew's conjecture. What analytic tools are needed to establish this conjecture?

APPENDIX. The appendix contains the proofs that were omitted in Sections 2 and 3.

Proof of (2.4). It follows immediately from the definition of a dual cone that $C \subset (C^*)^*$. The other containment depends on the fact that $C$ is closed. Suppose $x \in (C^*)^*$ and $x \notin C$. Since $C$ is closed, $\|x-E(x \mid C)\| > 0$. But, $x \in (C^*)^*$ and $x-E(x \mid C) \in C^*$ imply that $0 \geq (x,x-E(x \mid C)) = \|x-E(x \mid C)\|^2 + (E(x \mid C),x-E(x \mid C)) = \|x-E(x \mid C)\|^2$. This contradiction shows that $(C^*)^* \subset C$.

Proof of Lemma 2.2. We first note that if $E(x-\mu_0 \mid S_\mu) = b\mu$, then $E(x-\mu_0 \mid C_\mu) = b'\mu$ where $b' = b \vee 0$, and $E(x-\mu_0 \mid S_\mu) = E(x \mid S_\mu)$. Hence, $E(x-\mu_0 \mid C_\mu) = E(x \mid C_\mu)$ and so we establish (2.5) with $\mu_0 = 0$.

We consider the two cases $\mu \in C$ and $-\mu \in C^*$ separately. Suppose $\mu \in C$ and $0 \leq b \leq 2$. Using (2.3) followed by Lemma 2.1 and (2.3) again, we see that

$$\begin{aligned}\|E(x-bE(x \mid C_\mu) \mid C)\|^2 &= \|x-bE(x \mid C_\mu)\|^2 - \|E(x-bE(x \mid C_\mu) \mid C^*)\|^2 \\ &\leq \|x-bE(x \mid C_\mu)\|^2 - \|E(x \mid C^*)\|^2 \\ &= \|x\|^2 + b(b-2)\|E(x \mid C_\mu)\|^2 - \|E(x \mid C^*)\|^2 \\ &\leq \|x\|^2 - \|E(x \mid C^*)\|^2 = \|E(x \mid C)\|^2.\end{aligned}$$

If $\mu \in C^*$, then $-bE(x \mid C_\mu) \in C^*$ for all $b \geq 0$. Thus, by Lemma 2.1 and (2.4),

$$\|E(x-bE(x \mid C_\mu) \mid C)\| \leq \|E(x \mid C)\|$$

for all $b \geq 0$.

<u>Proof of Lemma 2.3</u>. Because of (2.3), $\|E(x+y \mid C)\|$ can be written as

$$\text{(A.1)} \quad \|E(E(x \mid C)+E(x \mid C^*) + E(y \mid C) + E(y \mid C^*) \mid C)\| = \|E(E(x \mid C) + E(y \mid C) + z \mid C)\|$$

where $z = E(x \mid C^*) + E(y \mid C^*) \in C^*$. Applying (2.4) and Lemma 2.1, (A.1) is bounded above by $\|E(E(x \mid C) + E(y \mid C) \mid C)\| = \|E(x \mid C) + E(y \mid C)\|$. The second inequality in Lemma 2.3 follows from the triangular inequality for norms.

<u>Proof of Lemma 2.4</u>. The first conclusion of part (a) follows from the third condition in (2.1) and the facts that $-v \in S$ whenever $v \in S$ and $S \subset C$. For the second conclusion in part (a), we check the three conditions in (2.1). Clearly, $E(x \mid C)-v \in C$ and $(x-v-(E(x \mid C)-v), E(x \mid C)-v)$ $= (x-E(x \mid C), E(x \mid C)) - (x-E(x \mid C), v)$, where the first term on the r.h.s is zero by (2.1) and the second term is zero because of the first part of (a).

For part (b), we assume $S$ is a closed subspace contained in $C$ and show that $E(x \mid S)$ satisfies the three conditions that characterize the projection of $E(x \mid C)$ onto $S$. Of course, $E(x \mid S) \in S$, $(E(x \mid C)-E(x \mid S), E(x \mid S)) = (x-E(x \mid S), E(x \mid S))-(x-E(x \mid C), E(x \mid S)) = 0$ (recall, $E(x \mid S), -E(x \mid S) \in S \subset C$), and for $u \in S$, $(E(x \mid C)-E(x \mid S), u)$ $= (x-E(x \mid S), u)-(x-E(x \mid C), u) = 0$ (again, $u, -u \in S \subset C$).

We prove part (d) before (c). So, we assume that $C \subset S$ and again verify the conditions in (2.1). By definition, $E(x \mid C) \in C$, and because $x-E(x \mid S) = E(x \mid S^{\perp})$, $(E(x \mid S)-E(x \mid C), E(x \mid C)) = (x-E(x \mid C), E(x \mid C))$ $- (x-E(x \mid S), E(x \mid C)) = -(E(x \mid S^{\perp}), E(x \mid C)) = 0$ since $C \subset S$. For $y \in C$, $(E(x \mid S)-E(x \mid C), y) = (x-E(x \mid C), y) - (E(x \mid S^{\perp}), y) = (x-E(x \mid C), y) \le 0$.

For part (c), assume that $S$ is a closed subspace contained in $C$. By part (a), $E(x \mid C)-E(x \mid S) = E(x-E(x \mid S) \mid C) = E(E(x \mid S^{\perp}) \mid C)$. By part (a), for $v \in S$, $(E(E(x \mid S^{\perp}) \mid C),v) = (E(x \mid S^{\perp}),v) = 0$ and so, $E(E(x \mid S^{\perp}) \mid C) \in C \cap S^{\perp}$. Hence, $E(E(x \mid S^{\perp}) \mid C) = E(E(x \mid S^{\perp}) \mid C \cap S^{\perp})$ and the latter is $E(x \mid C \cap S^{\perp})$ by part (d).

Proof of Lemma 2.9. For the proof of the first conclusion, we note that $x \in (FC)^{*I} \Leftrightarrow (x,y) \le 0$ for all $y \in FC \Leftrightarrow (x,Fz) \le 0$ for all $z \in C \Leftrightarrow (F^{-1}x,z)_W \le 0$ for all $z \in C \Leftrightarrow x \in FC^{*W}$.

For the second conclusion, we show that $FE_W(x \mid C)$ satisfies the three conditions that characterize $E(Fx \mid FC)$ (cf. (2.1)). Let $y = E_W(x \mid C)$. Of course, $Fy \in FC$, $(Fx-Fy,Fy) = (x-y,y)_W = 0$. and for $z \in FC$, $(Fx-Fy,z) = (x-y,F^{-1}z)_W \le 0$ since $F^{-1}z \in C$.

## REFERENCES

Abelson, R.P. and Tukey, J.W. (1963). Efficient utilization of nonnumeric information in quantitative analysis: general theory and the case of simple order. Ann. Math. Statist. 34, 1347-1369.

Anderson, T.W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Amer. Math. Soc. 6, 170-176.

Barlow, R.E., Bartholomew, D.J., Bremner, J.M. and Brunk, H.D. (1972). Statistical Inference Under Order Restrictions. Wiley, New York.

Bartholomew, D.J. (1959). A test for homogeneity for order alternatives, I, II. Biometrika 46, 36-48, 328-335.

Bartholomew, D.J. (1961). A test for homogeneity of means under restricted alternatives (with discussion). J. Roy. Statist. Soc. B18, 234-239.

Bartholomew, D.J. (1983). Isotonic inference, Encyclopedia of Statistical Sciences. John Wiley and Sons.

Dykstra, Richard L. and Robertson, Tim. On testing monotone tendencies. J. Amer. Statist. Assoc. 78, 342-350.

Lee, Chu-In Charles (1983). Statistical inference in normal model under linear constraints, Memorial University of Newfoundland Technical Report.

Lehmann, E.L. (1959). Testing Statistical Hypotheses. Wiley, New York.

Pillers, Carolyn, Robertson, Tim and Wright, F.T. (1984). A Fortran program for the level probabilities of order restricted inference. J. Roy. Statist. Soc. C33,

Robertson, Tim (1978). Testing for and against an order restriction on multinomial parameters. J. Amer. Statist. Assoc. 73, 197-202.

Robertson, Tim and Wegman, E.J. (1978). Likelihood ratio tests for order restrictions in exponential families. Ann. Statist. 6, 485-505.

Robertson, Tim and Wright, F.T. (1982). On measuring the conformity of a parameter set to a trend, with applications. Ann. Statist. 10, 1234-1245.

Robertson, Tim and Wright, F.T. (1983). On approximation of the level probabilities and associated distributions in order restricted inference. Biometrika 70, 597-606.

Robertson, Tim (1984). Monotone relationships, Encyclopedia of Statistical Sciences. John Wiley and Sons.

Schaafsma, W. and Smid, L.J. (1966). Most stringent somewhere most powerful tests against alternatives restricted by a number of linear inequalities. Ann. Math. Statist. 37, 1161-1172.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | | |
|---|---|---|
| 1. REPORT NUMBER<br>14 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>On the Power Functions of Test Statistics in Order Restricted Inference | | 5. TYPE OF REPORT & PERIOD COVERED<br>Technical Report<br>6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>Hari Mukerjee, Statistics, Univ. of Calif. Davis, CA<br>Tim Robertson, Statistics, Univ. of Iowa, Iowa City, IA<br>F.T. Wright, Math & Statistics, Univ. of Missouri, Rolla, MO | | 8. CONTRACT OR GRANT NUMBER(s)<br>N00014-80-0321<br>N00014-80-0322 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Office of Naval Research<br>Statistics and Probability Program-Code 436<br>Arlington, Virginia | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS | | 12. REPORT DATE<br>October, 1984<br>13. NUMBER OF PAGES<br>51 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>Unclassified<br>15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Order restricted tests, isotonic inference, power, likelihood ratio tests, contrast tests.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We study the power functions of both the likelihood ratio and contrast statistics for detecting a totally ordered trend in a collection of means of normal populations. Monotonicity properties are found and both radial limits and limits along lines parallel to the cone of points satisfying the trend are examined. An optimal contrast test for testing a trend as a null hypothesis is derived.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

FILMED

11-84

DTIC